ICALP 2026
July 7
London, UK

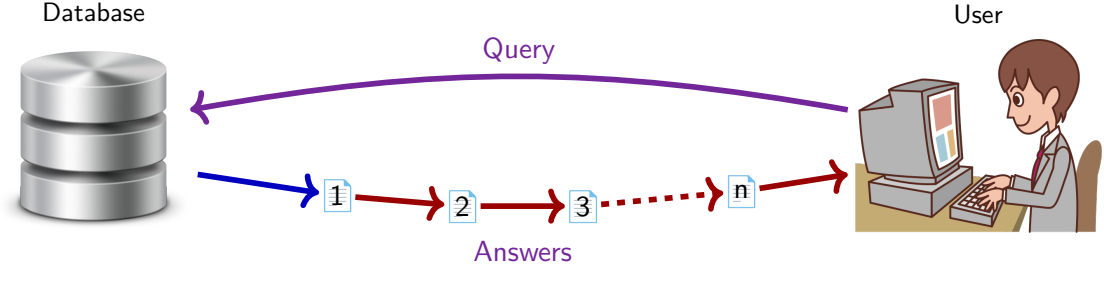
Gray Codes With Constant Delay and Constant Auxiliary Space

Antoine Amarilli [1]
Claire David [2]
Nadime Francis [2]
Victor Marsault [2]
Mikaël Monet [1]
Yann Strozecki [2,3]

1. Univ. Lille, Inria, CNRS, Centrale Lille, France
2. Univ. Gustave Eiffel, CNRS, France
3. Univ. de Versailles Saint-Quentin, France

Motivation

From graph databases...



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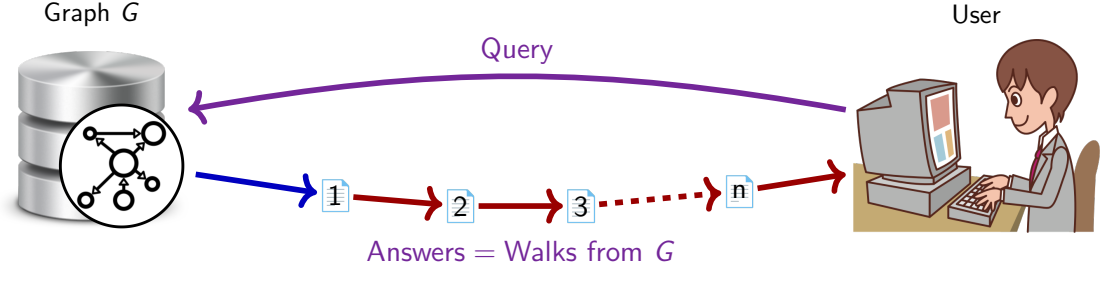

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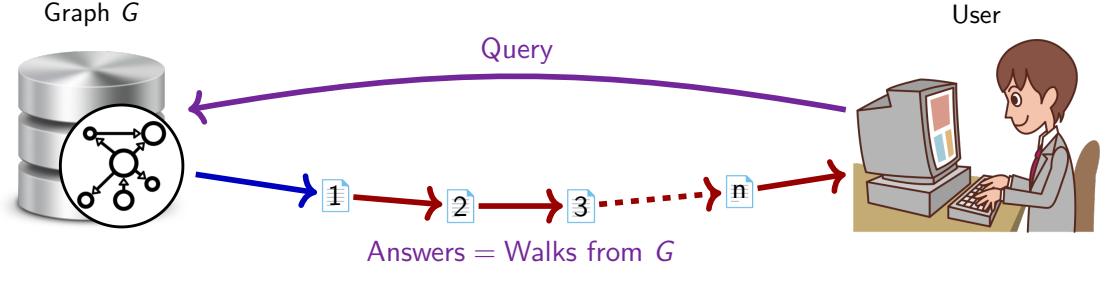

Situation:

- Some tasks have optimal solutions
- Can we reduce the output time w.r.t. the walk length ℓ ?

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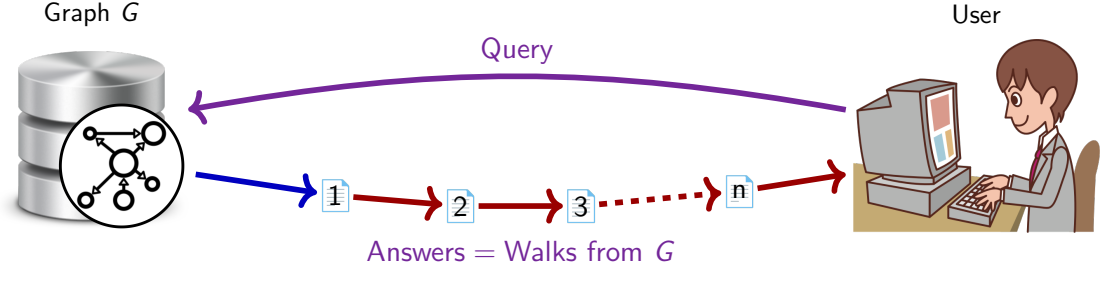
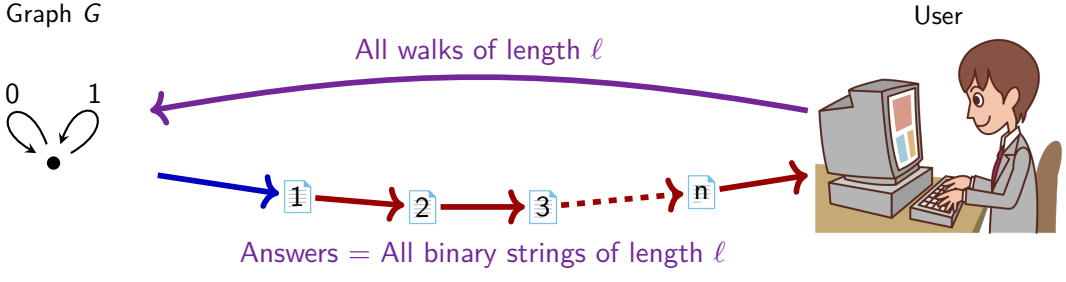


Situation:

- Some tasks have optimal solutions
- Can we reduce the output time w.r.t. the walk length ℓ ?

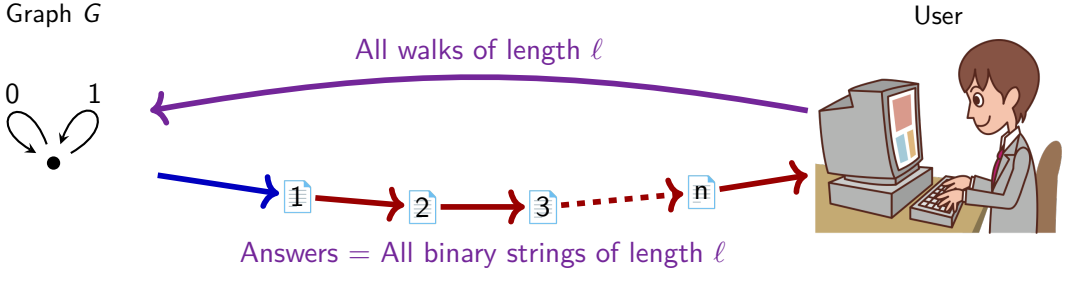
Our idea:

- Use Gray-code techniques
- I.e., output only the diff between walk with resources independent from ℓ


Graph G
User
All walks of length ℓ
0
1
1
2
3
n
Answers = All binary strings of length ℓ

Motivation

...to Gray codes



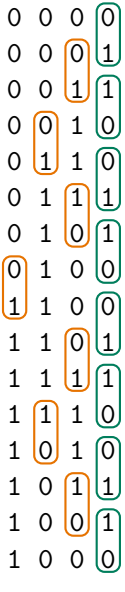
Our goal:

- Generate all binary strings of length ℓ...
- ...using resources independent from ℓ
  - Delay: time between consecutive output
  - Space (aside from the current output)

A **Gray code** defines
- for every length ℓ,
- a sequence of the 2^ℓ binary strings of length ℓ
- s.t. two successive strings differ by one bit.

Ex: Reflected Binary Gray Code (Gray 1947, Mutze 2023)
- At even steps, flip the rightmost bit
- At odd steps, flip the bit directly left of the rightmost 1.

RBGC



Constant Delay, Constant Auxiliary Space

A Gray code implementation has **constant delay** if
- the next bit to flip is computed in constant* time

* independent of ℓ

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- Constant-delay implementation: maintain a linked list of 1's.

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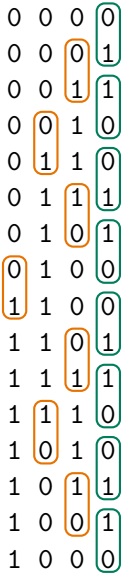

Constant Delay, Constant Auxiliary Space

A Gray code implementation has **constant delay** if
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A Gray code implementation uses **constant auxiliary space** if
- the next bit to compute requires constant* memory, aside from the current binary string.

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- Constant-delay implementation: maintain a linked list of 1's.
- Constant-aux-space implementation: scan for the rightmost 1.

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- Constant-delay implementation: maintain a linked list of 1's.
- Constant-aux-space implementation: scan for the rightmost 1.
- **No implementation of RBGC can be both.**

RBGC

"Turing machine with a fixed-sized tape"

A **tape machine** consists of

- A tape of **fixed** size ℓ with endmarkers $\triangleright$ and $\triangleleft$
- Finitely many states
- Output states, that produce the tape in constant time
- A head that reads/writes in a window of size n

Transitions look like: $\cdots 010 \cdots \rightsquigarrow \cdots 011 \cdots$

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Goal:

- Produce each binary string exactly once
- Description independent of the length ℓ
- The machine must be constant-delay

A **deque machine** is similar except:

- It works on **deque** of **fixed** size ℓ
- Transitions push/pop k characters on either ends

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Why deque machines ?

- It generalizes better to walks (our motivation)
  $\rightarrow$ Edge substitution in the middle is not very useful

General idea:
- Distribute strings of length ℓ in a complete binary tree
- Perform a DFS of that tree

Issue:
- How to keep delay constant?

→
- Classical *"Even-odd trick"*

Issue for tape machines:
- 3 bit flips between outputs
- $3 \neq 1$

→
- Encode parity on the tape
- Smart choice of first child

Issue for deque machines:
- How to stop?
- Obscure parity swap

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Either:
- Perform two DFS's
- Use *"lookahead"*

Technique overview

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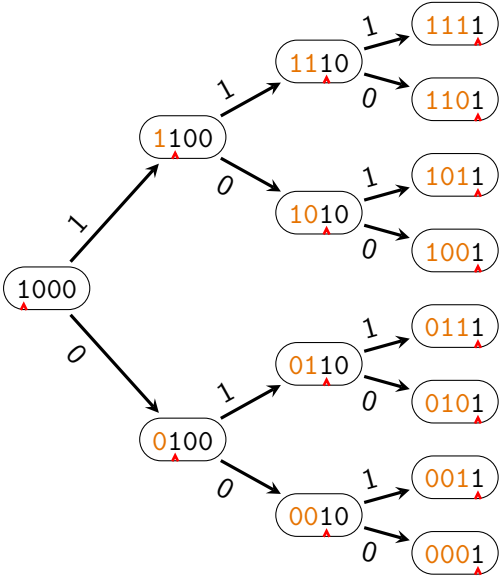
General idea ($\ell = 4$ in the example)

Building the tree

The string 0000 is treated separately

Other binary strings encode nodes
in the complete binary tree of depth $\ell - 1$

- Write each string as $u\underset{\wedge}{1}0^*$
- Place the head $\underset{\wedge}{\ }$ on the rightmost 1
- It encodes the node reached by reading u from the root



General idea ($\ell = 4$ in the example)

Navigating the tree with local transformations:

Reminder: $u\underset{\uparrow}{1}0^k$ reached by u from the root
$\triangleright$ and $\triangleleft$ are end markers

- Detect the root: $\triangleright\underset{\uparrow}{1}\cdots$
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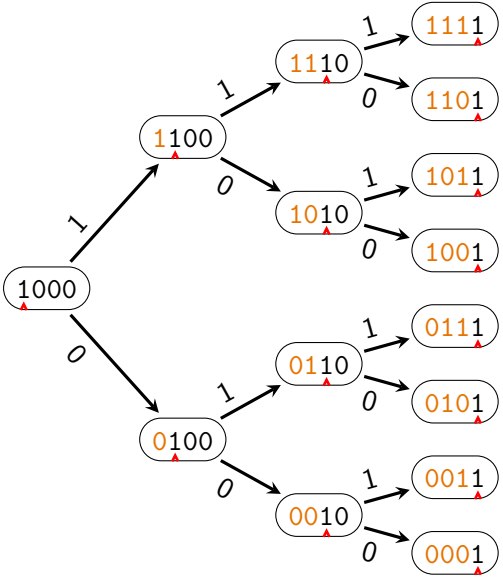

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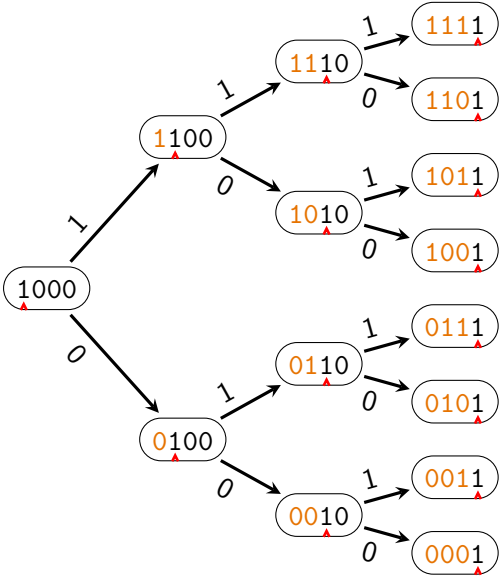

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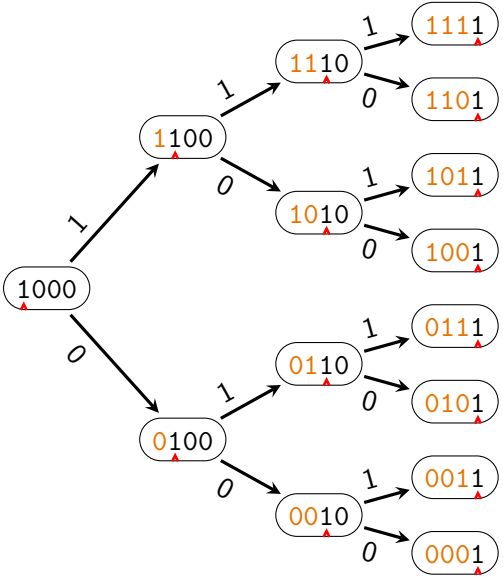

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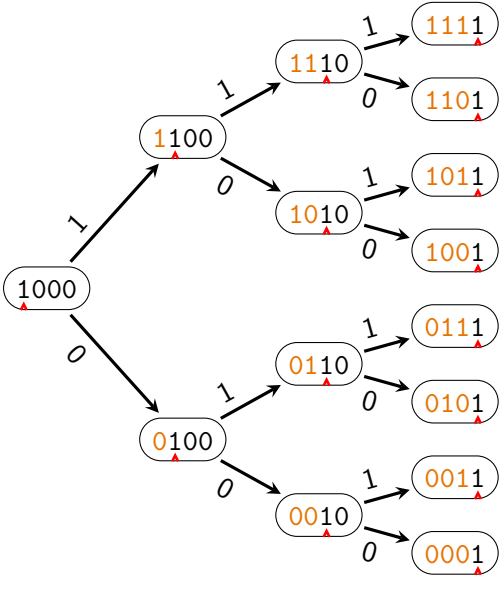
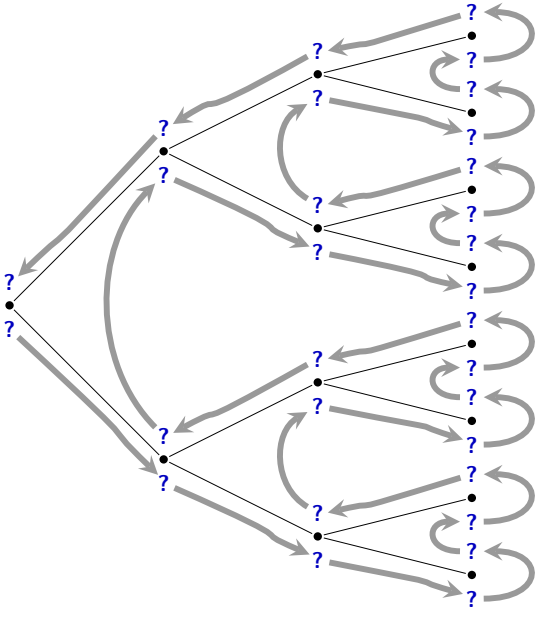
$\Rightarrow$ DFS implementable in a tape machine



Even-odd trick

Sekanina 1960; Karaganis 1968; Feder 1994

During a DFS, we want to
- output **once** at each node
- while each node is visited **twice**



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Naive approach

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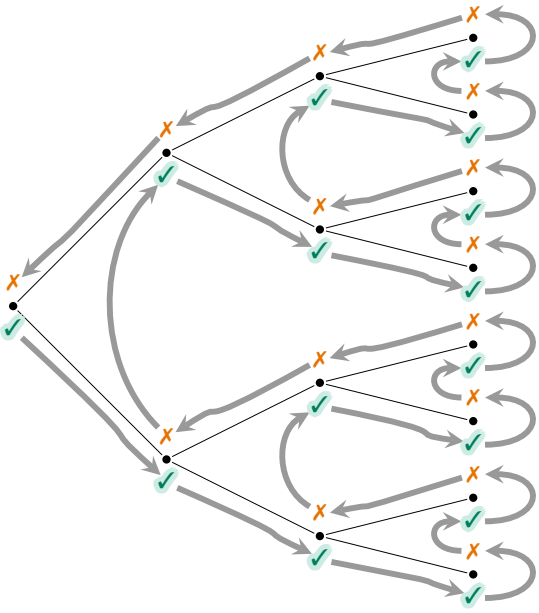

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Naive approach

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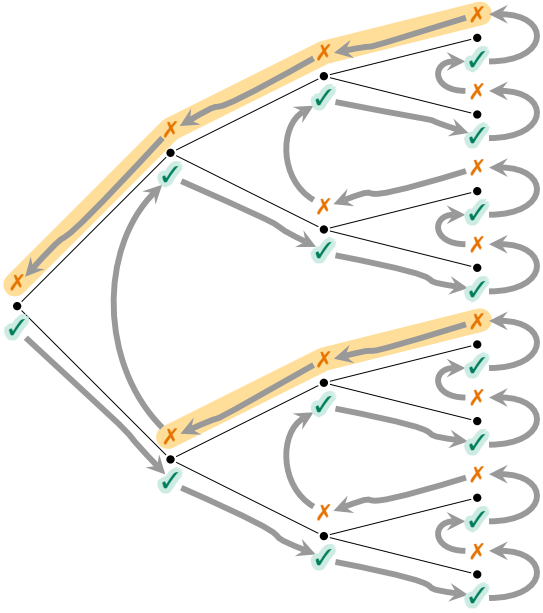
$\Rightarrow$ Long segments without output
$\Rightarrow$ In our case, linear delay



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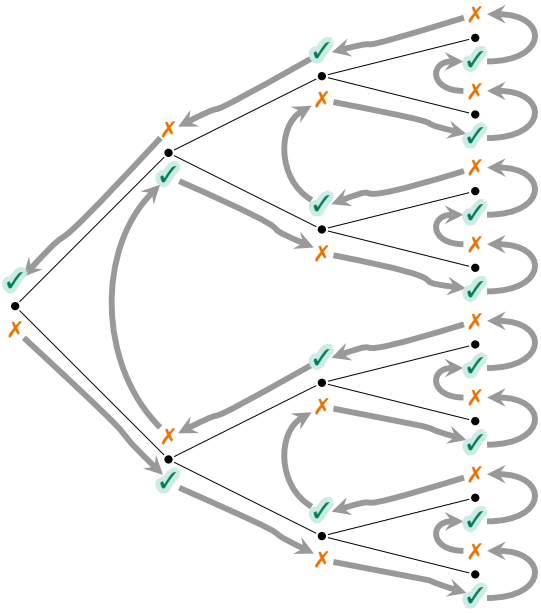
Even-odd trick

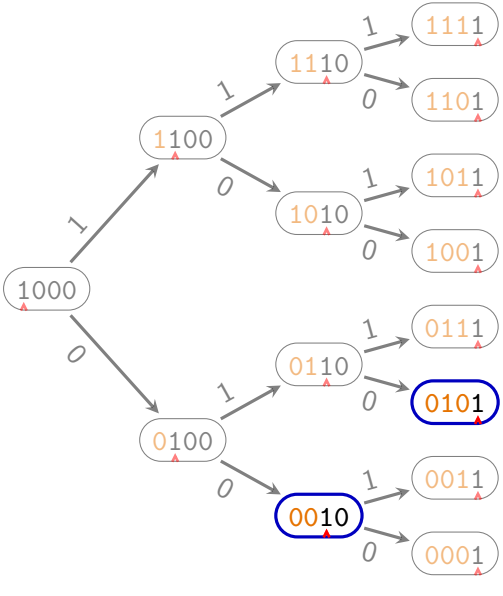
- **Odd** depth $\rightarrow$ when **entering** subtrees
- **Even** depth $\rightarrow$ when **leaving** subtrees



Not a Gray code (yet)

- Some navigation steps flip 2 bits
- The even-odd trick causes 2 steps between outputs

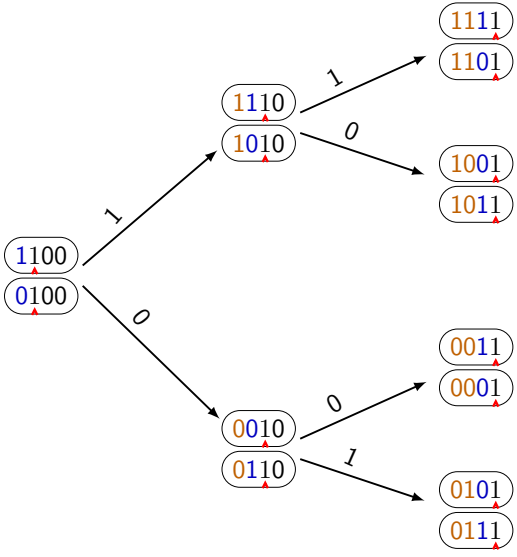
$\Rightarrow$ 3 flipped bits between some outputs



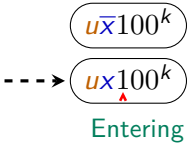
Writing the parity on the tape ($\ell = 4$ in the example)

We build the complete binary tree of depth $\ell - 2$

- Two strings in each node: $u\ 1\ 1\ 0^k$ / $u\ 0\ 1\ 0^k$
- Path from the root
- Parity bit



How the final tape machine works



where u: path from the root
x: parity bit

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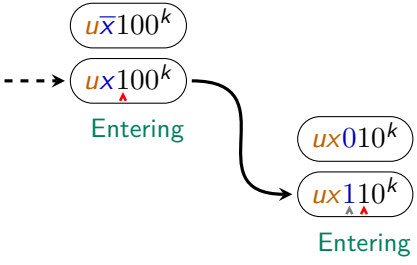

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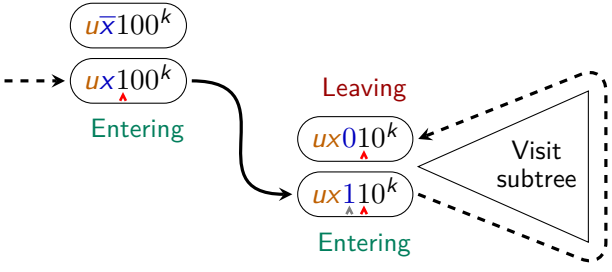

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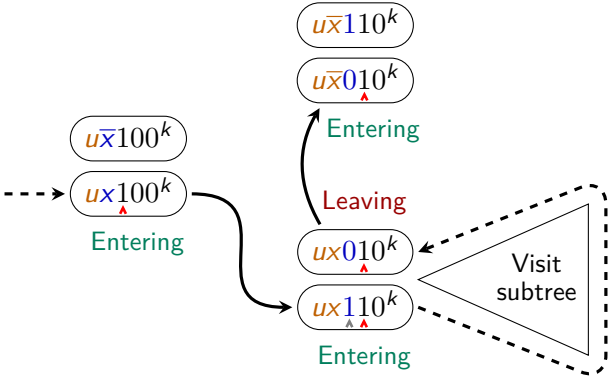

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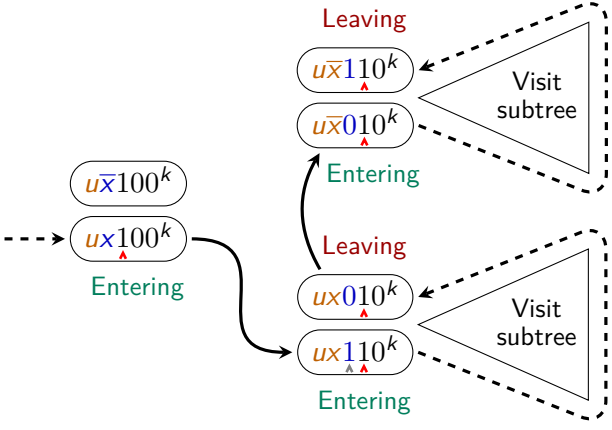

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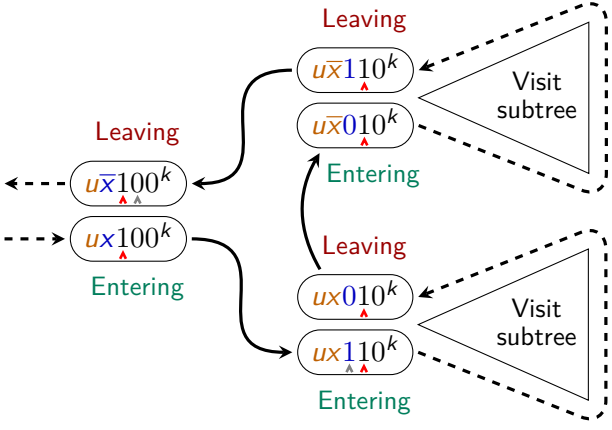

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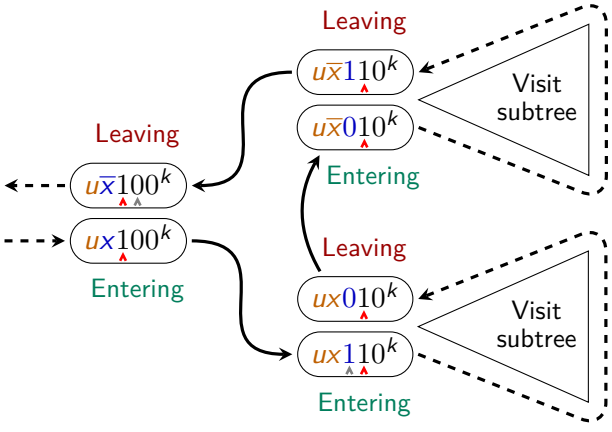

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The tape machine is simple: 2 states (Entering and Leaving) and a window of length 4

Contributions

1. Gray code implemented on a tape machine
2. Gray-like code implemented on a deque machine

Ongoing work

- Generalize 1 to non-binary alphabets
- Generalize 2 to strongly connected graphs

Future work

- Generalize 2 to all graphs
- Generalize 2 to more complex queries

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