Using automata to realise additions

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CNRS and Université Gustave-Eiffel

Special session in honor of Christiane Frougny's 75th birthday
Numeration 2023, Liège, Belgium

2023–05–23

1 Warm-up: Addition in integer base

Integer base

Definition

$p > 1$: an integer

$$\mathrm{Val}(a_n \cdots a_1 a_0) = a_n p^n + \cdots + a_1 p + a_0 = \sum_{i=0}^{n} a_i p^i$$

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Example: $p = 2$

$$\mathrm{Val}(0) = \mathrm{Val}(\varepsilon) = 0 \tag{1}$$

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$$\mathrm{Val}(1) = \mathrm{Val}(0000001) = 1 \tag{2}$$

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$$\mathrm{Val}(10) = \mathrm{Val}(02) = \mathrm{Val}(1\bar{1}0) = 2 \tag{3}$$

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Nonstandard digits

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$$\mathrm{Val}(0) = \mathrm{Val}(\varepsilon) = 0 \tag{1}$$

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Nonstandard digits

Negative digits

Addition is realised by a (multi-tape) automaton

$$\begin{array}{ccccccc} & 0 & 1 & 1 & 1 & 0 & 0 \\ + & 0 & 1 & 0 & 1 & 1 & 0 \\ = & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Addition is realised by a (multi-tape) automaton

$$
\begin{array}{ccccccc}
 & & & & & & \color{darkred}{0 \leftarrow \text{carry}} \\
 & 0 & 1 & 1 & 1 & 0 & 0 \\
+ & 0 & 1 & 0 & 1 & 1 & 0 \\
= & . & . & . & . & . & .
\end{array}
$$

Addition is realised by a (multi-tape) automaton

$$
\begin{array}{cccccccl}
 & & & & & {\color{darkred}0} & {\color{darkred}0} & {\color{darkred}\leftarrow \text{carry}} \\
 & 0 & 1 & 1 & 1 & 0 & 0 & \\
+ & 0 & 1 & 0 & 1 & 1 & 0 & \\
= & \cdot & \cdot & \cdot & \cdot & \cdot & 0 &
\end{array}
$$

Addition is realised by a (multi-tape) automaton

$$\begin{array}{cccccccl} & & & & 0 & 0 & 0 & \leftarrow \text{carry} \\ & 0 & 1 & 1 & 1 & 0 & 0 & \\ + & 0 & 1 & 0 & 1 & 1 & 0 & \\ = & \cdot & \cdot & \cdot & \cdot & 1 & 0 & \end{array}$$

Addition is realised by a (multi-tape) automaton

$$
\begin{array}{ccccccccl}
 & & & 1 & 0 & 0 & 0 & \leftarrow \text{carry} \\
 & 0 & 1 & 1 & 1 & 0 & 0 & \\
+ & 0 & 1 & 0 & 1 & 1 & 0 & \\
= & \cdot & \cdot & \cdot & 0 & 1 & 0 &
\end{array}
$$

Addition is realised by a (multi-tape) automaton

$$\begin{array}{ccccccc} & & \color{darkred}{1} & \color{darkred}{1} & \color{darkred}{0} & \color{darkred}{0} & \color{darkred}{0} \leftarrow \text{carry} \\ & 0 & 1 & 1 & 1 & 0 & 0 \\ + & 0 & 1 & 0 & 1 & 1 & 0 \\ = & \cdot & \cdot & 0 & 0 & 1 & 0 \end{array}$$

Addition is realised by a (multi-tape) automaton

$$
\begin{array}{cccccccl}
 & 1 & 1 & 1 & 0 & 0 & 0 & \leftarrow \text{carry} \\
 & 0 & 1 & 1 & 1 & 0 & 0 & \\
+ & 0 & 1 & 0 & 1 & 1 & 0 & \\
= & \cdot & 1 & 0 & 0 & 1 & 0 &
\end{array}
$$

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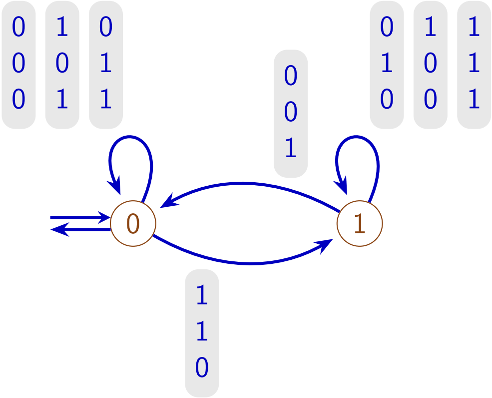
$$
\begin{array}{ccccccccl}
 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \leftarrow \text{carry} \\
 & & 0 & 1 & 1 & 1 & 0 & 0 & \\
+ & & 0 & 1 & 0 & 1 & 1 & 0 & \\
= & & 1 & 1 & 0 & 0 & 1 & 0 &
\end{array}
$$

Addition is realised by a (multi-tape) automaton

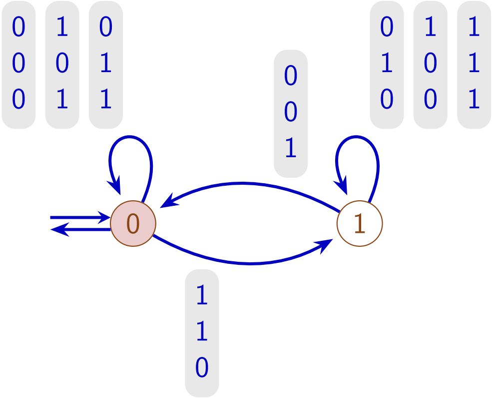
$$\begin{array}{ccccccccc}
 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \leftarrow \text{state} \\
 & & 0 & 1 & 1 & 1 & 0 & 0 & \\
+ & & 0 & 1 & 0 & 1 & 1 & 0 & \\
= & & 1 & 1 & 0 & 0 & 1 & 0 &
\end{array}$$



Addition is realised by a (multi-tape) automaton

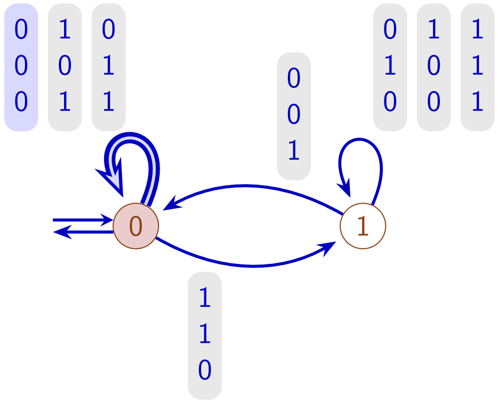
0 ← state

$$\begin{array}{ccccccc} & 0 & 1 & 1 & 1 & 0 & 0 \\ + & 0 & 1 & 0 & 1 & 1 & 0 \\ = & 1 & 1 & 0 & 0 & 1 & 0 \end{array}$$



Addition is realised by a (multi-tape) automaton

					0	0	← state
	0	1	1	1	0	0	
+	0	1	0	1	1	0	
=	1	1	0	0	1	0	



Addition is realised by a (multi-tape) automaton

				0	0	0	← state
	0	1	1	1	0	0	
+	0	1	0	1	1	0	
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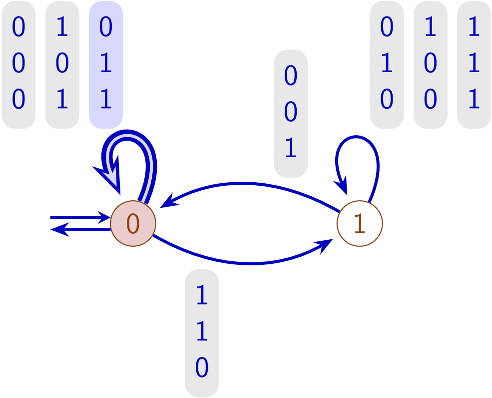
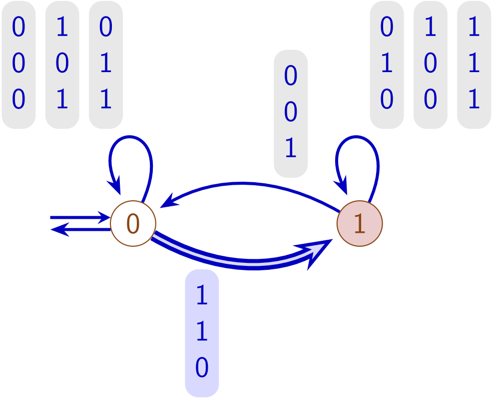


Addition is realised by a (multi-tape) automaton

$$\begin{array}{cccccccl}
 & & & 1 & 0 & 0 & 0 & \leftarrow \text{state} \\
 & 0 & 1 & 1 & 1 & 0 & 0 & \\
+ & 0 & 1 & 0 & 1 & 1 & 0 & \\
= & 1 & 1 & 0 & 0 & 1 & 0 &
\end{array}$$



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 & & 1 & 1 & 0 & 0 & 0 & \leftarrow & \text{state} \\
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\end{array}
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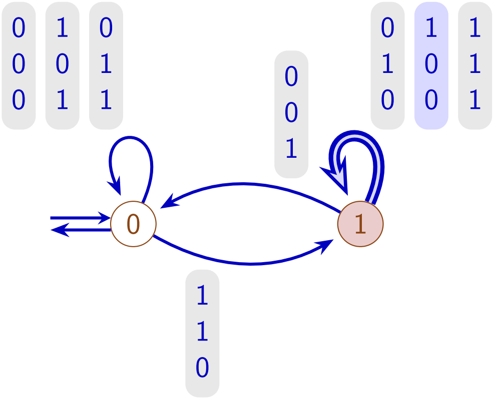

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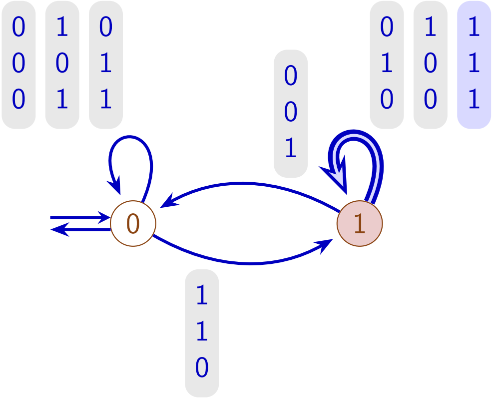
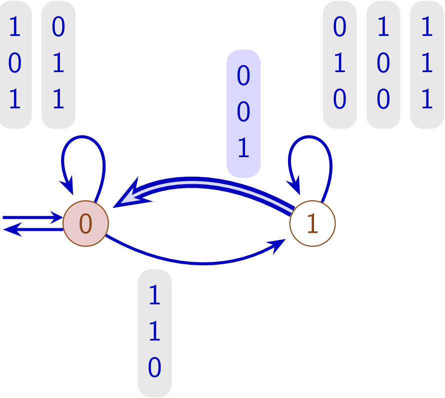
$$
\begin{array}{ccccccccl}
 & {\color{darkred}1} & {\color{darkred}1} & {\color{darkred}1} & {\color{darkred}0} & {\color{darkred}0} & {\color{darkred}0} & \leftarrow & \text{state} \\
 & 0 & 1 & 1 & 1 & 0 & 0 & & \\
+ & 0 & 1 & 0 & 1 & 1 & 0 & & \\
= & 1 & 1 & 0 & 0 & 1 & 0 & &
\end{array}
$$



Addition is realised by a (multi-tape) automaton

	0	1	1	1	0	0	0	← state
		0	1	1	1	0	0	
$+$		0	1	0	1	1	0	
$=$		1	1	0	0	1	0	



Addition is just a *Digit-conversion*

Addition

	0	1	1	1	0	0
$+$	0	1	0	1	1	0
$=$	1	1	0	0	1	0

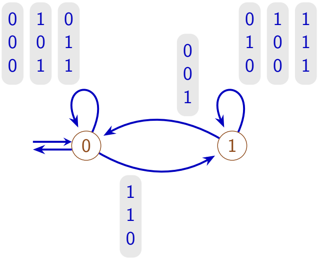


Figure: Additioner

Addition is just a *Digit-conversion*

Addition

	0	1	1	1	0	0
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$=$	1	1	0	0	1	0

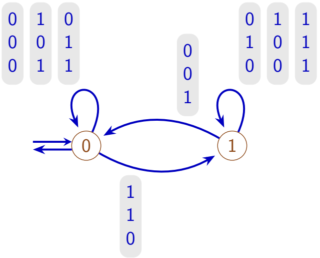

Figure: Additioner

Digit-conversion $\{0,1,2\} \to \{0,1\}$

0	2	1	2	1	0
1	1	0	0	1	0

Same value

Addition is just a *Digit-conversion*

Addition

$$\left.\begin{array}{cccccccc} & 0 & 1 & 1 & 1 & 0 & 0 \\ + & 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right\} \text{carry-less addition}$$
$$\begin{array}{cccccccc} = & 1 & 1 & 0 & 0 & 1 & 0 \end{array}$$

Digit-conversion $\{0,1,2\} \rightarrow \{0,1\}$

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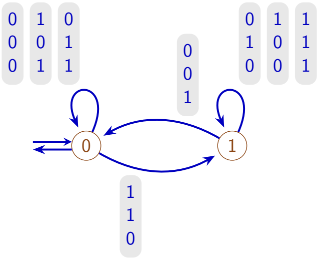
Same value



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carry-less addition

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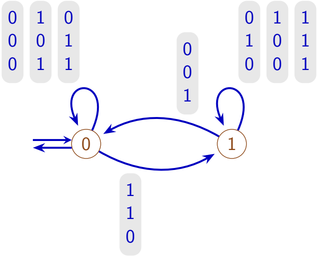

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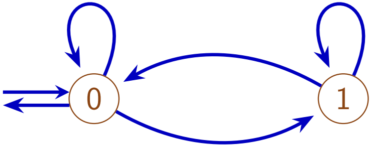

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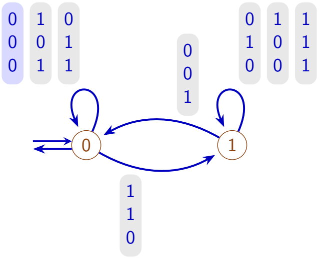

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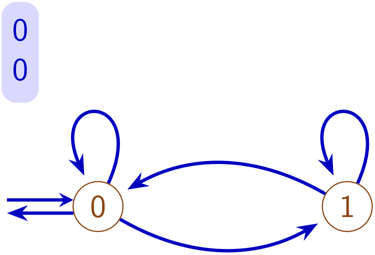

Figure: Converter $\{0, 1, 2\} \to \{0, 1\}$

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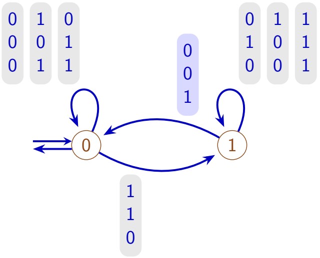
Addition

$$
\left.\begin{array}{cc}
 & 0\ 1\ 1\ 1\ 0\ 0 \\
+ & 0\ 1\ 0\ 1\ 1\ 0
\end{array}\right\}
\xrightarrow{\text{carry-less addition}}
\begin{array}{c}
0\ 2\ 1\ 2\ 1\ 0 \\
1\ 1\ 0\ 0\ 1\ 0
\end{array}
\quad \text{Same value}
$$

$$
\begin{array}{cc}
= & 1\ 1\ 0\ 0\ 1\ 0
\end{array}
$$

Digit-conversion $\{0,1,2\} \to \{0,1\}$



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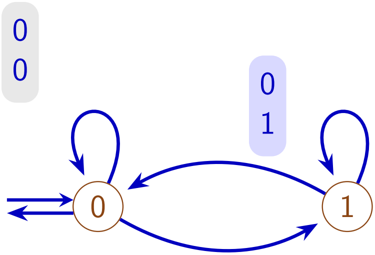

Figure: Converter $\{0,1,2\} \to \{0,1\}$

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Addition

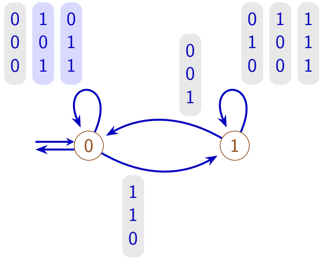
	0	1	1	1	0	0
$+$	0	1	0	1	1	0
$=$	1	1	0	0	1	0

carry-less addition

Digit-conversion $\{0,1,2\} \to \{0,1\}$

0	2	1	2	1	0
1	1	0	0	1	0

Same value



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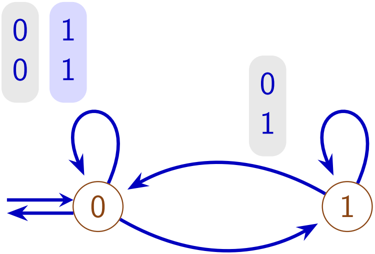

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\end{array}
$$

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$$
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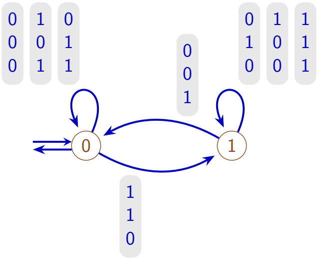
\end{array}
$$

Same value



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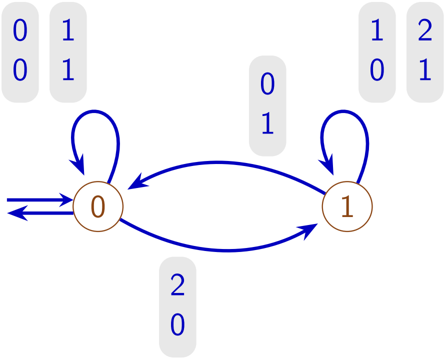

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One strong theorem based the additioner

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Büchi-Bruyère Theorem [Büchi'60][Bruyère'85][Charlier'18]

R : a relation in $\mathbb{N}^d$.
R is realised by a d-tape automaton in base p
$\iff$ R is definable by a formula in $FO[\mathbb{N}, +, V_p]$ †

†: $FO[\mathbb{N}, +, V_p]$ is the first-order logic with functions $+$ and V_p.
V_p is the function $n \mapsto 2^k$, the greater power of 2 such that $\frac{n}{2^k} \in \mathbb{N}$.

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V_p is the function $n \mapsto 2^k$, the greater power of 2 such that $\frac{n}{2^k} \in \mathbb{N}$.

$\Rightarrow$ Many properties are decidable for automatic sequences (periodicity, squarefreeness, etc.)

2 Journey from evaluator to additioner

Base β

$\beta > 1$: a real number

Evaluation in base β

Before the radix point

$$\mathrm{Val}(a_k \cdots a_1 a_0 \bullet) = \sum_{i=0}^{k} a_i \beta^i \tag{4}$$

After the radix point

$$\mathrm{Val}(\bullet a_1 \cdots a_k \cdots) = \sum_{i=0}^{\infty} a_i \beta^{-i} \tag{5}$$

$\beta > 1$: a real number

Evaluation in base β

Before the radix point

$$\mathrm{Val}(a_k \cdots a_1 a_0 \bullet) = \sum_{i=0}^{k} a_i \beta^i \tag{4}$$

After the radix point

$$\mathrm{Val}(\bullet a_1 \cdots a_k \cdots) = \sum_{i=0}^{\infty} a_i \beta^{-i} \tag{5}$$

In this talk, we do not care about representation (Greedy, normalisation, ...)

The evaluator $\mathcal{E}_A$

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- Edges: $s \xrightarrow{a} t \iff s\beta + a = t$

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$$\mathsf{Val}(a_k \cdots a_1 a_0 \,_\bullet) = \beta \times \mathsf{Val}(a_k \cdots a_1 \,_\bullet) + a_0$$

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$$\overbrace{\mathsf{Val}(a_k \cdots a_1 a_0\,\bullet)}^{x=} = \beta \times \overbrace{\mathsf{Val}(a_k \cdots a_1\,\bullet)}^{y=} + a_0$$

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- Edges: $s \xrightarrow{a} t \iff s\beta + a = t$

$$\overbrace{\mathrm{Val}(a_k \cdots a_1 a_0 \bullet)}^{x=} = \beta \times \overbrace{\mathrm{Val}(a_k \cdots a_1 \bullet)}^{y=} + a_0$$

$\implies$ $\mathcal{E}_A$ features the transition $x \xrightarrow{a_0} y$

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$\mathcal{E}_A$ is the infinite labelled graph defined by:

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- Edges: $s \xrightarrow{a} t \iff s\beta + a = t$

$$\overbrace{\mathrm{Val}(a_k \cdots a_1 a_0 \bullet)}^{x=} = \beta \times \overbrace{\mathrm{Val}(a_k \cdots a_1 \bullet)}^{y=} + a_0$$

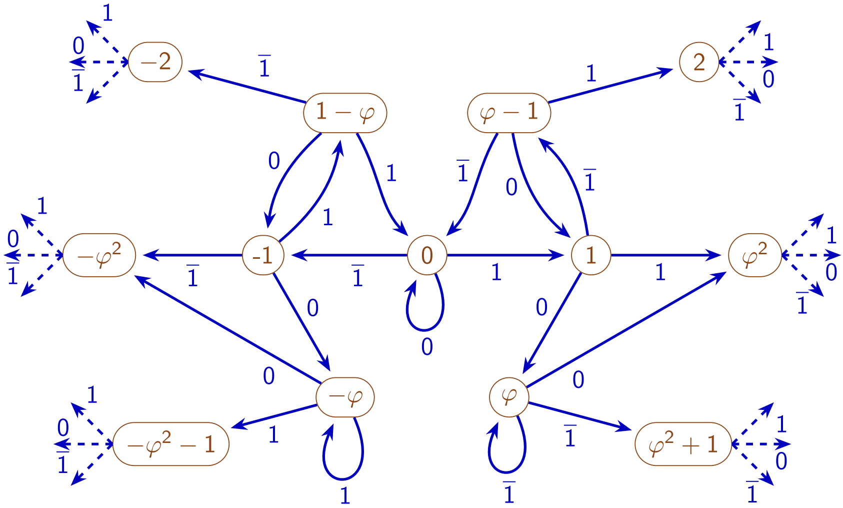
$$\implies \mathcal{E}_A \text{ features the transition } x \xrightarrow{a_0} y$$

Lemma

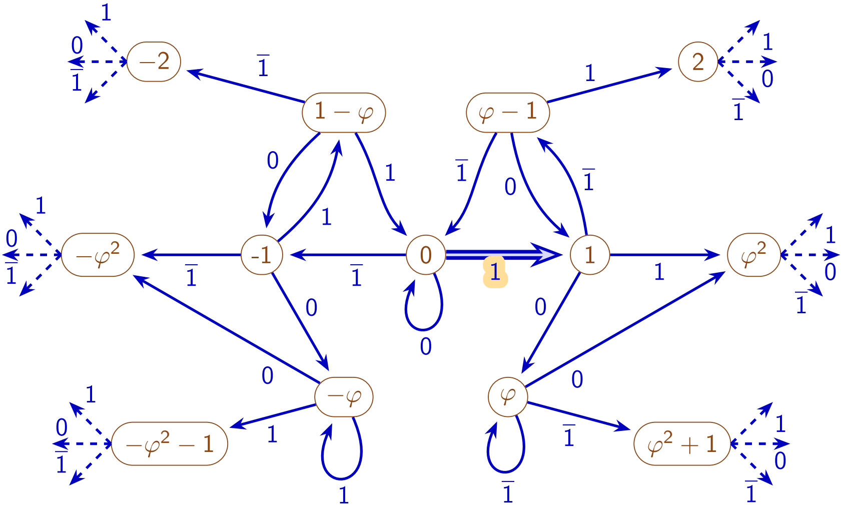
For each $a_k \cdots a_1 a_0$,

$$0 \xrightarrow{a_k} \cdots \xrightarrow{a_1} \xrightarrow{a_0} s \qquad \text{where } s = \mathrm{Val}(a_k \cdots a_1 a_0 \bullet) \tag{6}$$

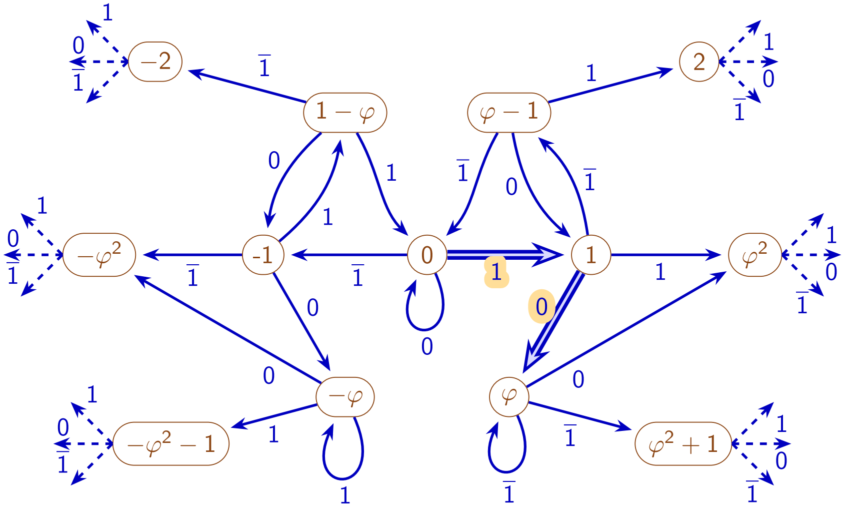
$\mathcal{E}_A$ with $A = \{\overline{1}, 0, 1\}$ and $\beta = \varphi = \frac{1+\sqrt{5}}{2}$



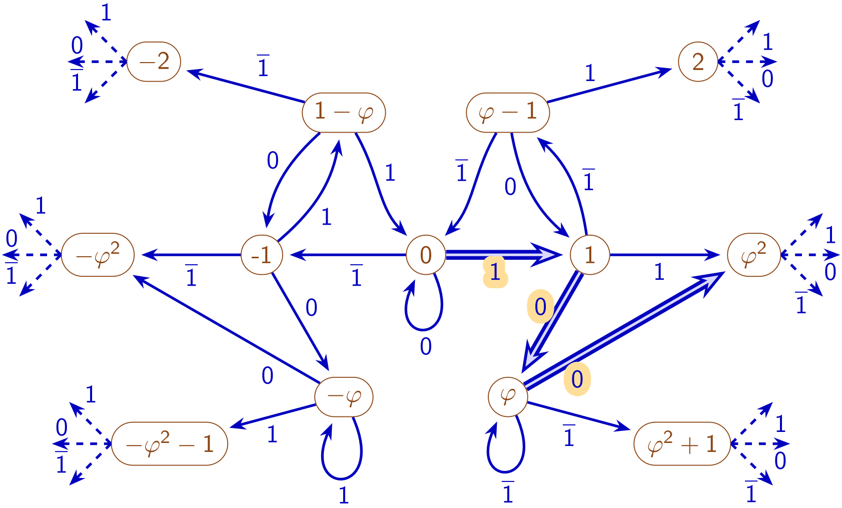
$\mathcal{E}_A$ with $A = \{\overline{1}, 0, 1\}$ and $\beta = \varphi = \frac{1+\sqrt{5}}{2}$



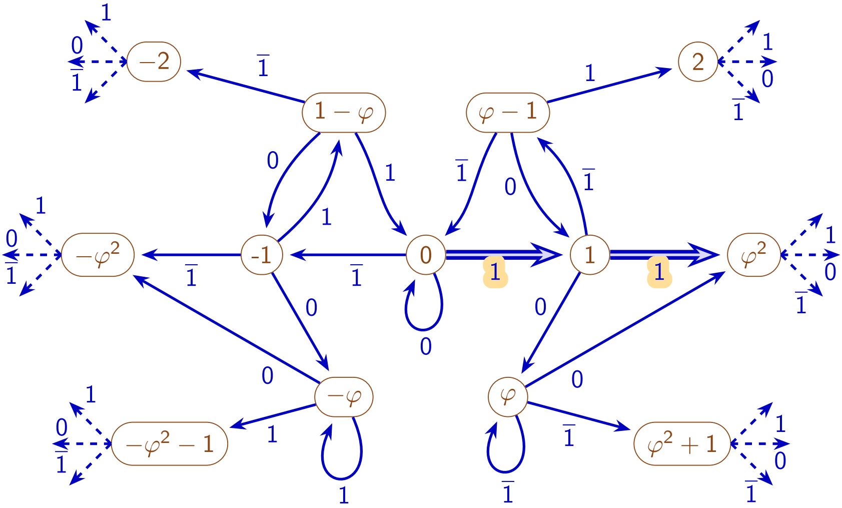
$\mathrm{Val}(1\bullet) = 1$



$\mathsf{Val}(10\bullet) = \varphi$

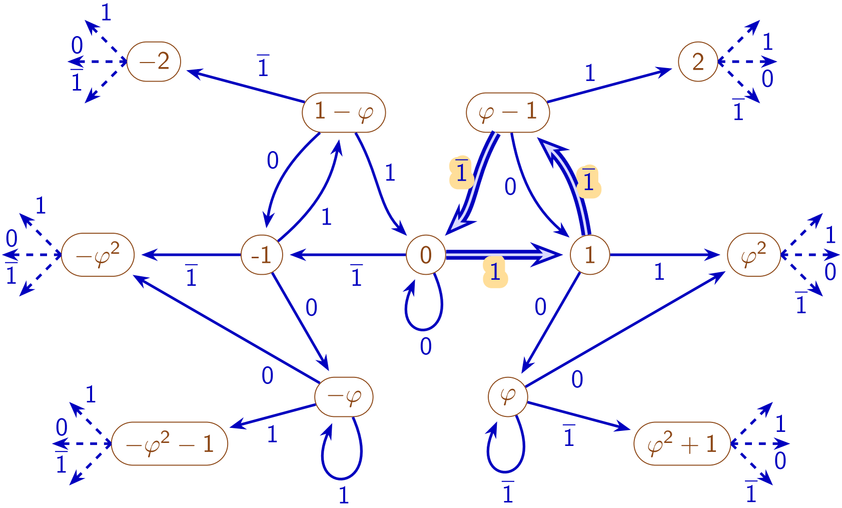


$$\mathrm{Val}(100\bullet) = \varphi^2$$



$\mathrm{Val}(11\bullet) = \varphi + 1 = \varphi^2$

$\mathcal{E}_A$ with $A = \{\overline{1}, 0, 1\}$ and $\beta = \varphi = \frac{1+\sqrt{5}}{2}$



$\mathrm{Val}(1\overline{1}\overline{1}\bullet) = \varphi^2 - \varphi - 1 = 0$

The zero-automaton $\mathcal{Z}_A$

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$\mathcal{Z}_A$ is defined as $\mathcal{E}_A$ except:

- $\mathcal{Z}_A$ is a Büchi automaton
- States: vertices of $\mathcal{E}_A$ in $\left[-\frac{d}{\beta-1}, \frac{d}{\beta-1}\right]$, where $d = \max_{a \in A}\{|a|\}$
- 0 is initial
- All states are final

$\mathcal{Z}_A$ accepts infinite words that labels walks starting from 0.

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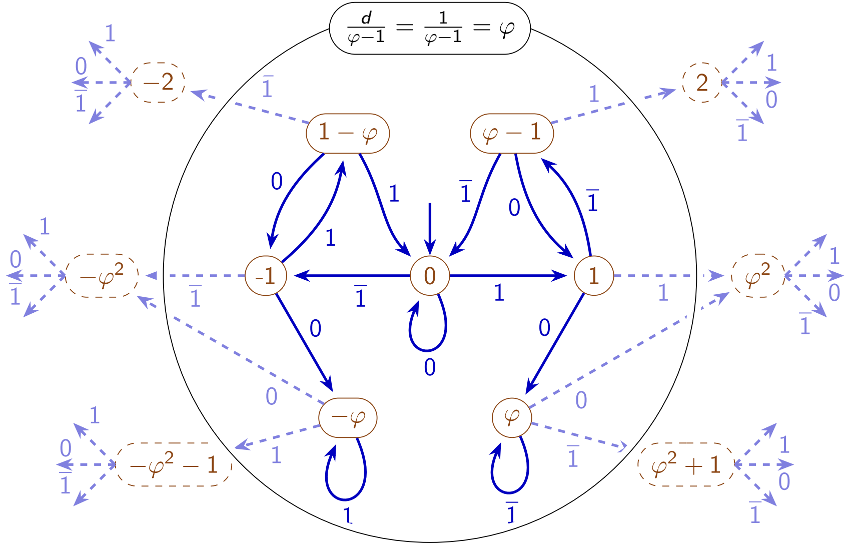
Theorem [Frougny'92]

$(a_0 a_1 \cdots a_k \cdots) \in A^{\mathbb{N}}$ is accepted by $\mathcal{Z}_A$

$$\iff \mathrm{Val}(\bullet a_0 a_1 \cdots a_k \cdots) = 0$$

$\mathcal{E}_A \to \mathcal{Z}_A$ with $A = \{\overline{1}, 0, 1\}$ and $\beta = \varphi = \frac{1+\sqrt{5}}{2}$



Zero-automaton Theorem

The following are equivalent.

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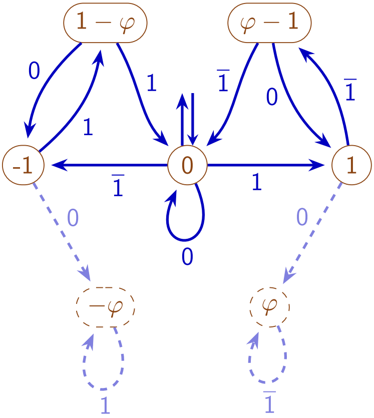
Corollary

$a_k \cdots a_1 a_0$ accepted by $\mathcal{F}_A \iff \mathrm{Val}(a_k \cdots a_1 a_0 \bullet) = 0$



$1-\varphi$
$\varphi-1$
-1
0
1
$-\varphi$
φ
0
1
1
1
$\bar{1}$
0
$\bar{1}$
$\bar{1}$
1
0
0
0
1
$\bar{1}$

Converter $\mathcal{C}_{D_1,D_2}$

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Behavior we want

Accept pairs $\begin{array}{ll} a_k \cdots a_1 a_0 & \in D_1 \\ b_k \cdots b_1 b_0 & \in D_2 \end{array}$ such that

$$\mathrm{Val}(a_k \cdots a_1 a_0 \bullet) = \mathrm{Val}(b_k \cdots b_1 b_0 \bullet)$$

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$$\begin{aligned} & \mathrm{Val}(a_k \cdots a_1 a_0 \bullet) = \mathrm{Val}(b_k \cdots b_1 b_0 \bullet) \\ \iff & \mathrm{Val}(a_k \cdots a_1 a_0 \bullet) - \mathrm{Val}(b_k \cdots b_1 b_0 \bullet) = 0 \\ \iff & \mathrm{Val}\left((a_k - b_k) \cdots (a_1 - b_1)(a_0 - b_0) \bullet\right) = 0 \end{aligned}$$

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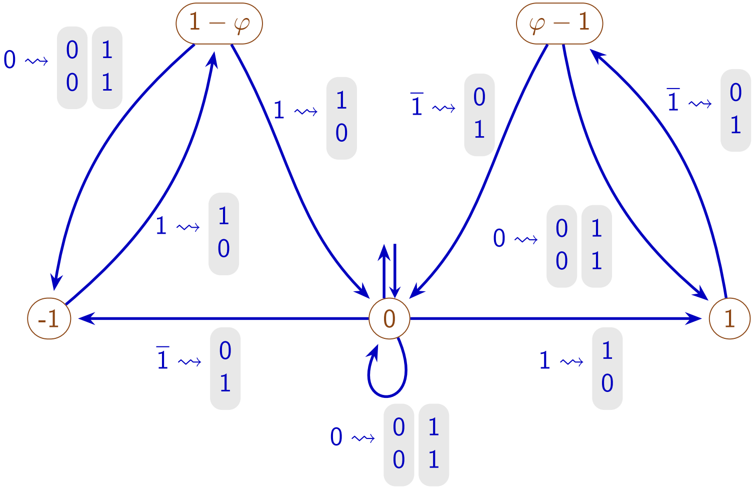
&\iff (a_k - b_k) \cdots (a_1 - b_1)(a_0 - b_0) \text{ is accepted by } \mathcal{F}_A
\end{aligned}$$

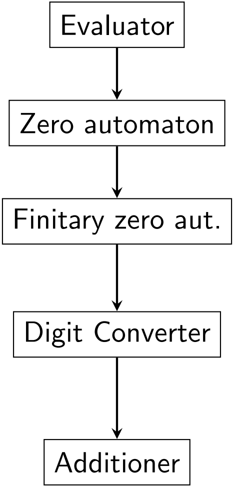
where $A = \{(d_1 - d_2) \mid d_1 \in D_1,\ d_2 \in D_2\}$

$\mathcal{F}_A \to \mathcal{C}_{D_1,D_2}$ with $A = \{\bar{1}, 0, 1\}$ and $D_1 = D_2 = \{0, 1\}$



$$a \in \{\bar{1}, 0, 1\} \rightsquigarrow \left\{ \begin{matrix} d_1 \in \{0,1\} \\ d_2 \in \{0,1\} \end{matrix} \;\middle|\; a = (d_1 - d_2) \right\}$$

Recap: from evaluator to additioner



Recap: from evaluator to additioner

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Evaluator	$a_k \cdots a_1 a_0$	$0 \xrightarrow{a_k} \cdots \xrightarrow{a_1}\xrightarrow{a_0} s$ where $s = \mathrm{Val}(a_k \cdots a_1 a_0 \bullet)$
$\downarrow$		
Zero automaton		
$\downarrow$		
Finitary zero aut.		
$\downarrow$		
Digit Converter		
$\downarrow$		
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↓		
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$U = (U_k)_{k \in \mathbb{N}}$ a Pisot linear system

A statement similar to the Büchi-Bruyère Theorem

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Theorem [Bruyère-Hansel'97]

R : a relation in $\mathbb{N}^d$.

R is realised by a d-tape automaton in U

$\iff$ R is definable by a formula in $FO[\mathbb{N}, +, V_U]$ †

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V_U is the function $n \mapsto U_k$, where k is the rightmost 1 in the representation of n

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$\Rightarrow$ Many properties are decidable for U-automatic sequences (periodicity, squarefreeness, etc.)

3 Foreshadowing: Rational base

Evaluation in base $\frac{p}{q}$

Before the radix point

$$\mathrm{Val}(a_n \cdots a_1 a_0 \bullet) = \sum_{i=0}^{n} a_i \left(\frac{p}{q}\right)^i \tag{7}$$

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Theorem [Akiyama-Frougny-Sakarovitch'08]

$\forall n \in \mathbb{N}$,

$$\exists a_k \cdots a_1 \in \{0, \ldots, p-1\} \qquad \mathrm{Val}(a_k \cdots a_1 a_0 \bullet) = n \tag{8}$$

Base $\frac{p}{q}$ features an additioner

Theorem [Akiyama-Frougny-Sakarovitch'08]

Addition is realised by a finite automaton in base $\frac{p}{q}$.

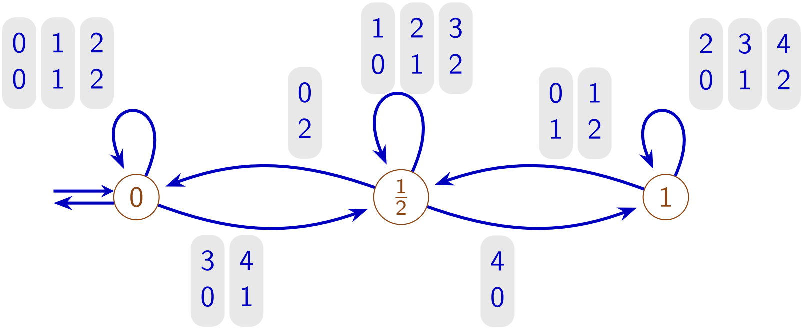

Figure: Converter $\{0, 1, 2, 3, 4\} \to \{0, 1, 2\}$ in base $\frac{3}{2}$

Wait? Rational numbers are **not** Pisot numbers

β is a Pisot number iff

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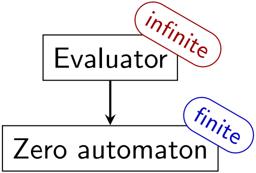
Explanation of the non-contradiction

If β is Pisot



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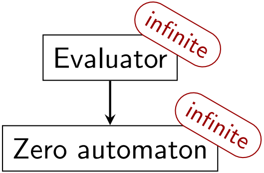

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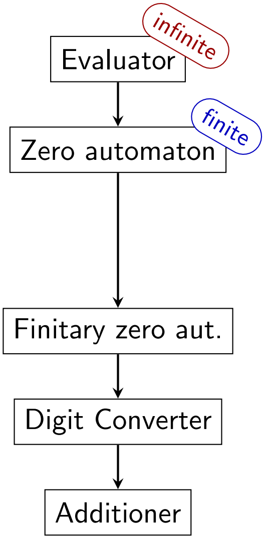

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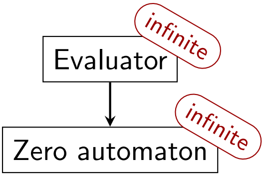

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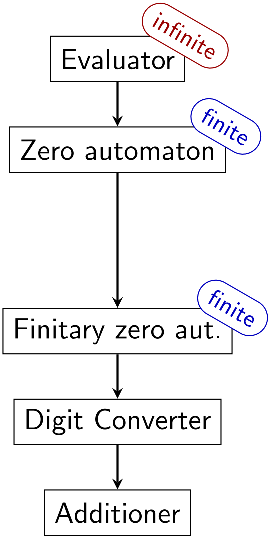

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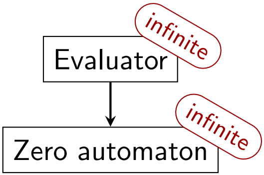

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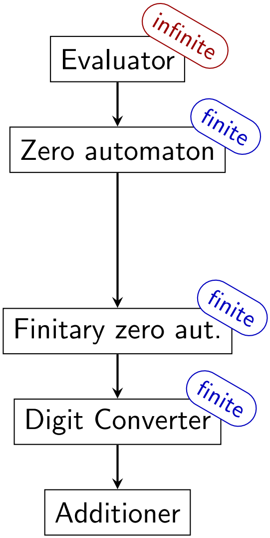

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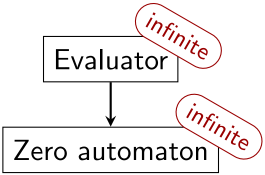

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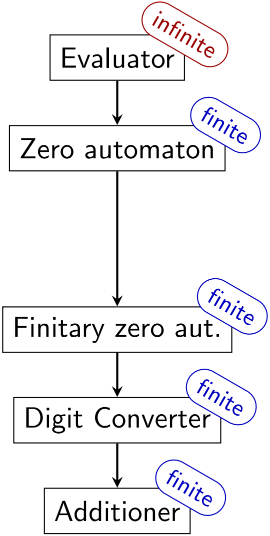

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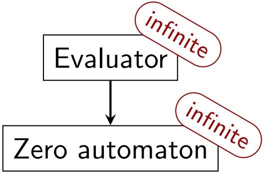

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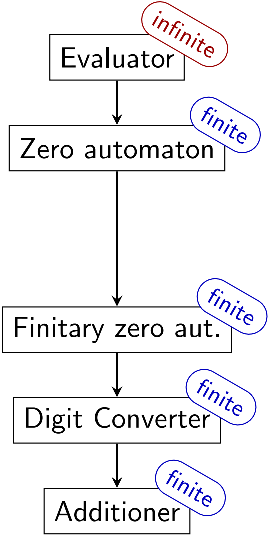

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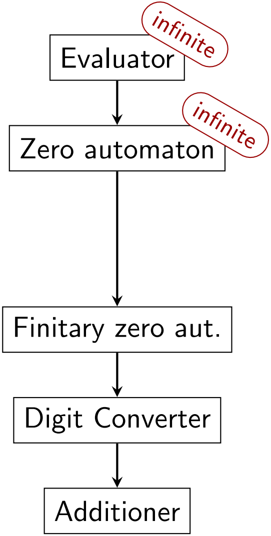

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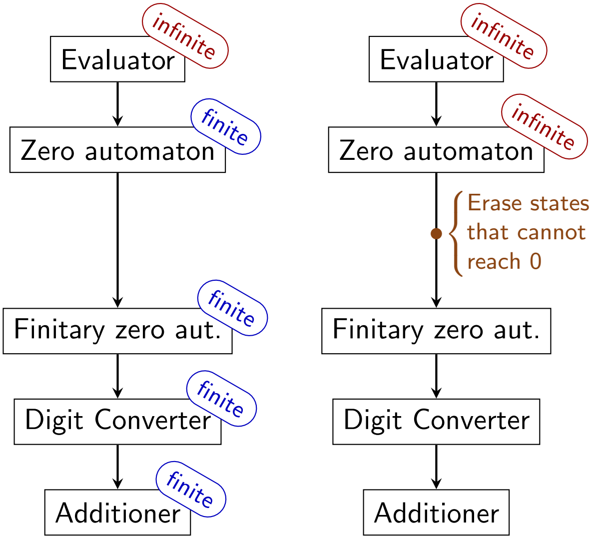

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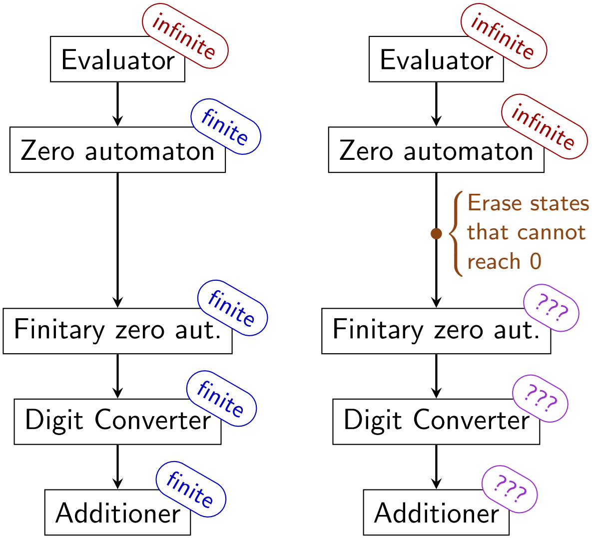

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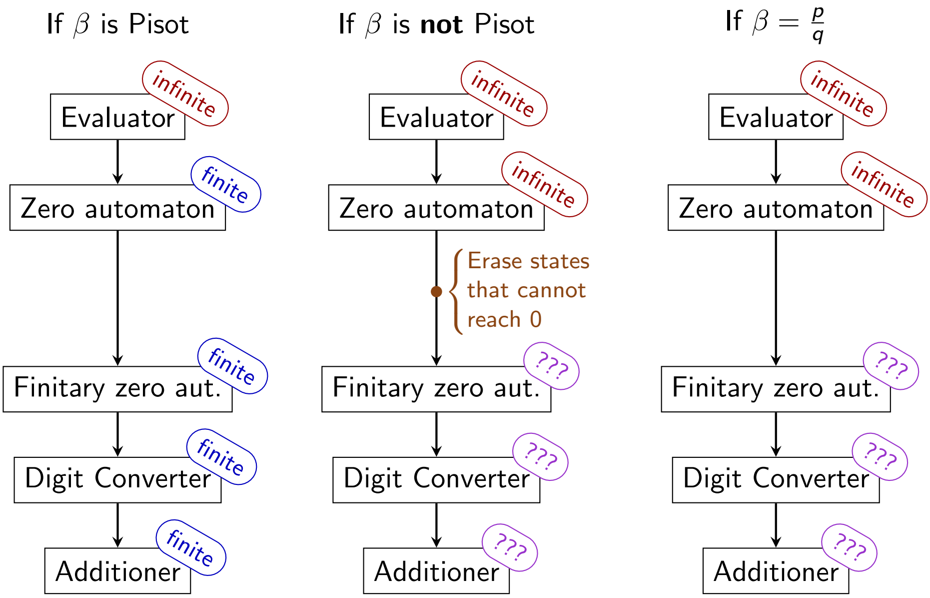

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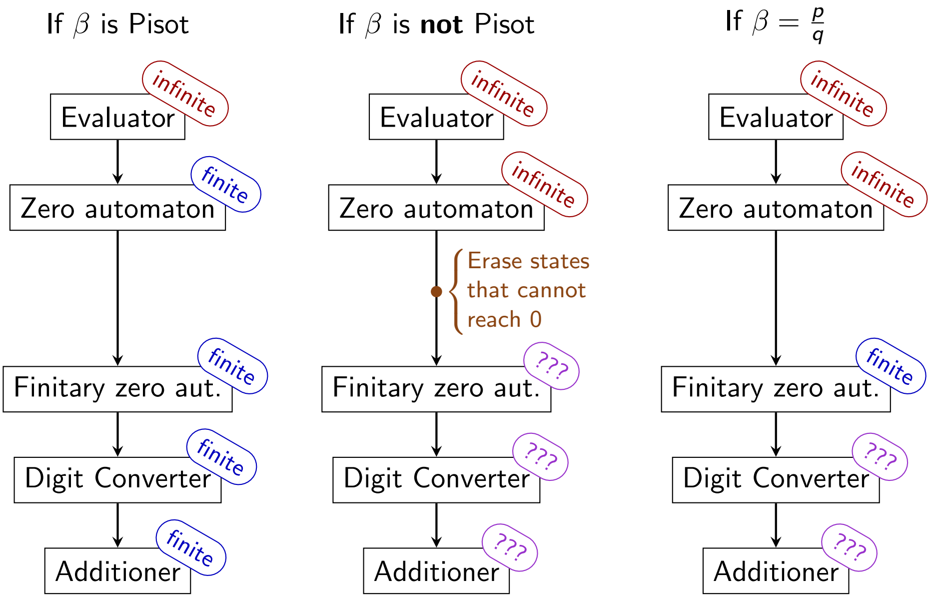

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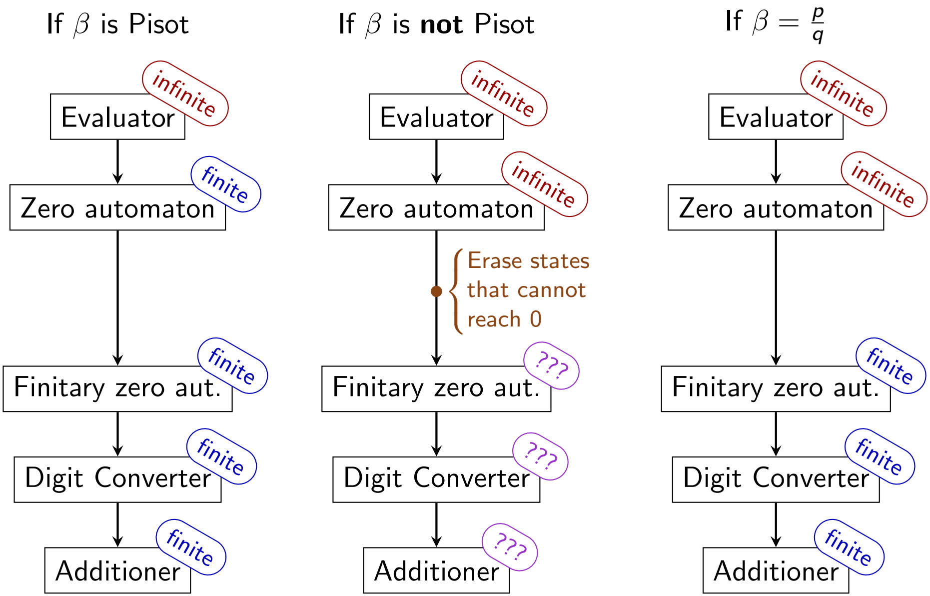

Some statement similar to the Büchi-Bruyère Theorem

$N_{\frac{p}{q}} = \mathrm{Val}(A^*)$ with $A = \{0, \ldots, p-1\}$

Theorem [M.'21]

R : a relation in $(N_{\frac{p}{q}})^d$.

R is realised by a d-tape automaton in base $\frac{p}{q}$

$\iff$ R is definable by a formula in $FO[N_{\frac{p}{q}}, +, V_{\frac{p}{q}}]$ †

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What about $\frac{p}{q}$-automatic sequences?

$\rightarrow$ Mostly open

Recap: from evaluator to additioner

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