An efficient algorithm to decide periodicity of b-recognisable sets using MSDF convention

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Plan

Integer base numeration systems

- $b > 1$
- *Alphabet used to represent numbers:* $\{0, 1, \ldots, b-1\}$

- $$\begin{array}{lccc} \textsc{val} & : & \{0,1,\ldots,b-1\}^* & \longrightarrow & \mathbb{N} \\ & & d_n \cdots d_1 d_0 & \longmapsto & d_n b^n + \cdots + d_1 b^1 + d_0 b^0 \end{array}$$

In base $b = 2$, $\textsc{val}(010011) = 0 + 2^3 + 0 + 0 + 2^1 + 2^0 = 19$.

- $$\begin{array}{lcccc} \textsc{rep} & : & \mathbb{N} & \longrightarrow & \{0,1,\ldots,b-1\}^* \\ & & 0 & \longmapsto & \varepsilon \\ & & n > 0 & \longmapsto & \textsc{rep}(m)\, d \ , \end{array}$$

  where (m, d) *is the Eucl. div of* n *by* b*.*

In base $b = 2$, $\textsc{rep}(19) = \textsc{rep}(9)\,1 = \textsc{rep}(4)\,11 = \cdots = 10011$.

b-recognisable sets

Definition

X: a set of integers.

*X is b-**recognisable** if $\text{REP}(X)$ is a regular language.*

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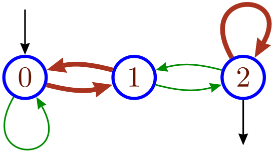
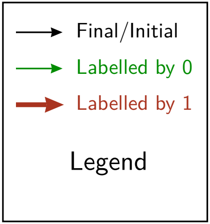
Theorem (folklore)

- *Each eventually-periodic set is b-recognisable.*



Automaton accepting
$0^*\text{REP}(2+3\mathbb{N})$



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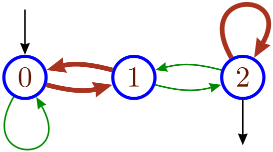
- *Each eventually-periodic set is b-recognisable.*
- *Some sets are 2-recognisable but not 3-recognisable.*



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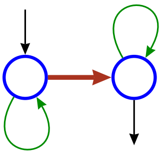
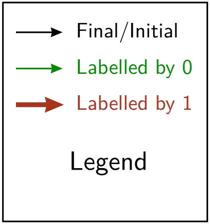


Automaton accepting
$0^*\textsc{rep}\big(\{2^i \mid i \in \mathbb{N}\}\big)$



b-recognisable sets (2)

Theorem (Cobham, 1969)

b, c : *two integer bases, multiplicatively independent*[†].
X : *a set of integers.*

$$\left.\begin{array}{l} X \text{ is } b\text{-recognisable} \\ X \text{ is } c\text{-recognisable} \end{array}\right\} \Longrightarrow X \text{ is eventually periodic}$$

[†]such that $b^i \neq c^j$ for all $i, j > 0$.

Corollary

$$\left\{ \text{Eventually periodic sets} \right\} = \left\{ \text{Sets } b\text{-recognisable for all } b \right\}$$

Periodicity

- ***Parameter**: an integer base $b > 1$.*
- ***Input**: a deterministic finite automaton $\mathcal{A}$*
  (hence the b-recognisable set X accepted by $\mathcal{A}$).
- ***Question**: is X eventually periodic ?*

The Periodicity problem (2)

First answers

Theorem (Honkala, 1986)

PERIODICITY *is decidable.*

Theorem (Muchnik, 1991)

A generalisation of PERIODICITY *is decidable in triple-exponential time.*

The PERIODICITY problem (3)

First efficient algorithms uses LSDF convention

Least Significant Digit First (LSDF) *: the input automaton reads its entry from right to left.*

Theorem (Leroux, 2005)

With LSDF convention, a generalisation of PERIODICITY *is decidable in polynomial time.*

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Remark

Making an automaton reads from right to left
requires a transposition and a determinisation
$\Rightarrow$ *Exponential blow-up*

Note (Allouche Rampersad Shallit, 2009)

PERIODICITY *is decidable in exponential time.*

Theorem (M.-Sakarovitch, 2013)

With LSDF convention, PERIODICITY *is decidable in linear time if the input automaton is minimal.*

Theorem

PERIODICITY *is decidable in* $O(b\,n\,\text{log}(n))$ *time*

(where n *is the state-set cardinal.)*

Plan

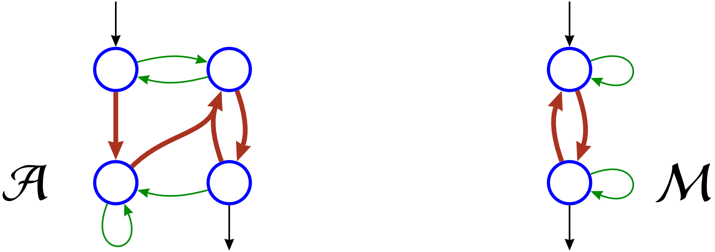
Definition

$\mathcal{A}, \mathcal{M}$: two complete DFA
φ: a function {states of $\mathcal{A}$} $\to$ {states of $\mathcal{M}$}

φ is a ***pseudo-morphism*** *$\mathcal{A} \to \mathcal{M}$ if*

- *φ maps the initial state of $\mathcal{A}$ to the initial state of $\mathcal{M}$*
- $s \xrightarrow{d} s'$ *in $\mathcal{A}$* $\implies$ $\varphi(s) \xrightarrow{d} \varphi(s')$ *in $\mathcal{M}$*

(A pseudo-morphism is a morphism with no condition on final states.)



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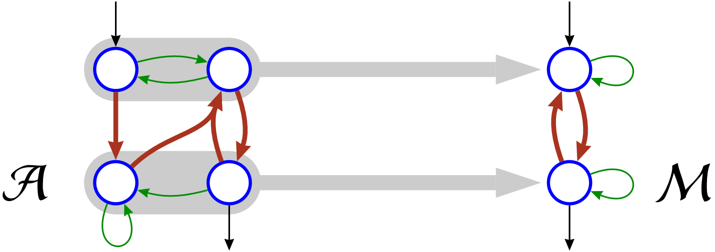

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$\mathcal{A}, \mathcal{M}$: *two complete DFA*
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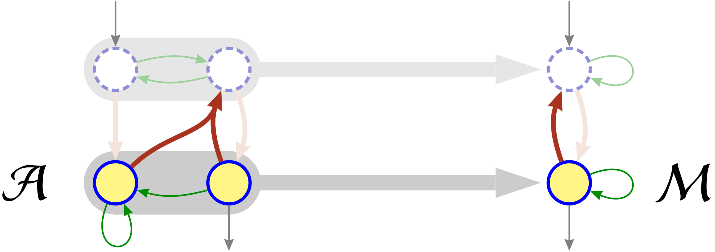

Pseudo-morphism (1) – Definition & Example

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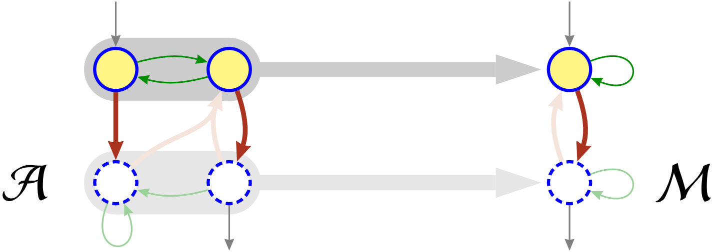

Pseudo-morphism (2) – Computation

Lemma

$\mathcal{A}, \mathcal{M}$: two complete DFA
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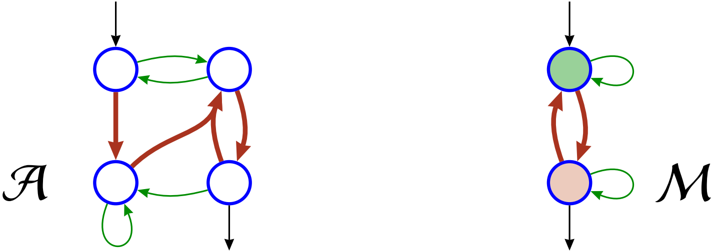
The pseudo-morphism $\varphi : \mathcal{A} \to \mathcal{M}$, if it exists, can be computed in $O(bn)$ time.



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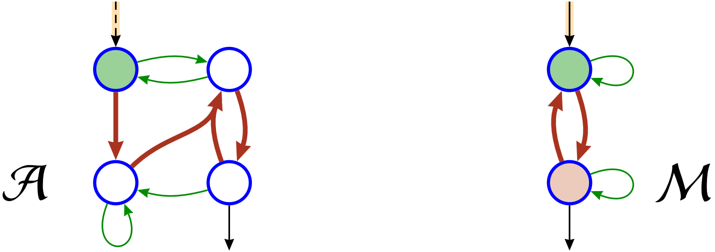

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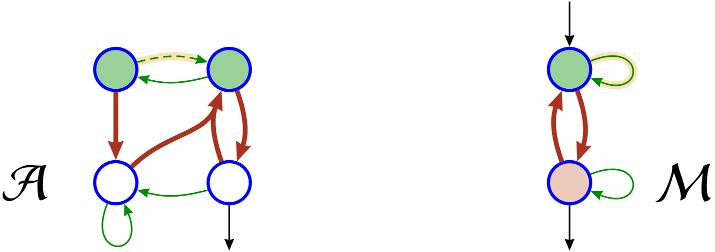

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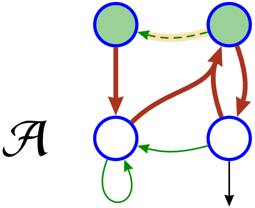

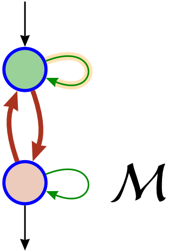

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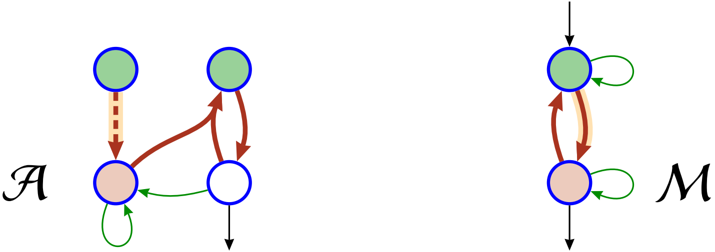

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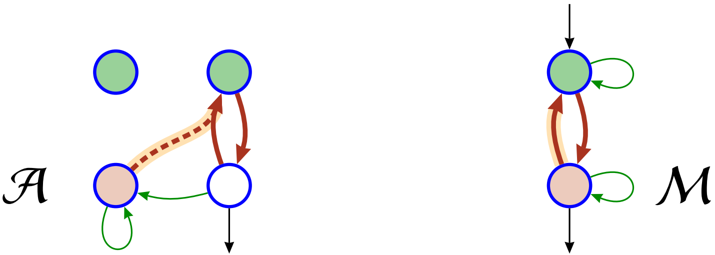

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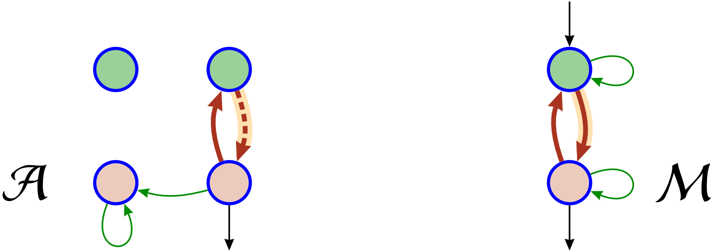

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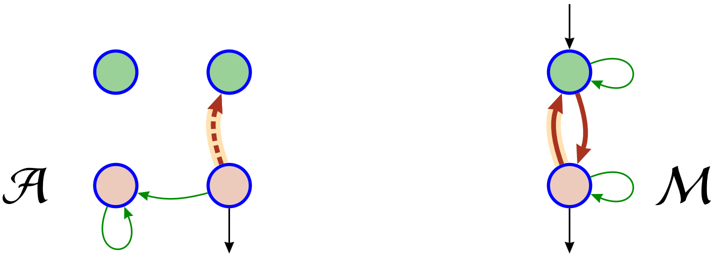

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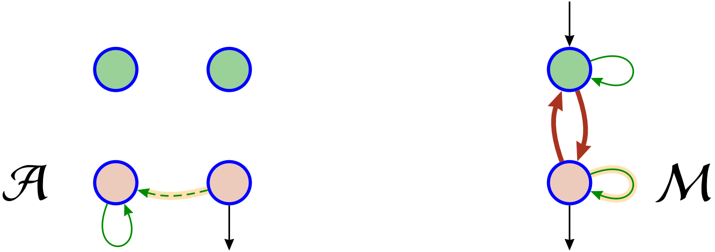

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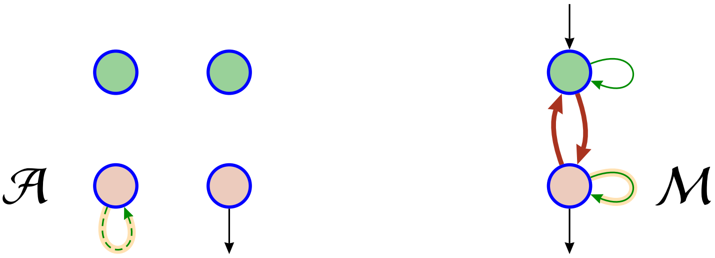

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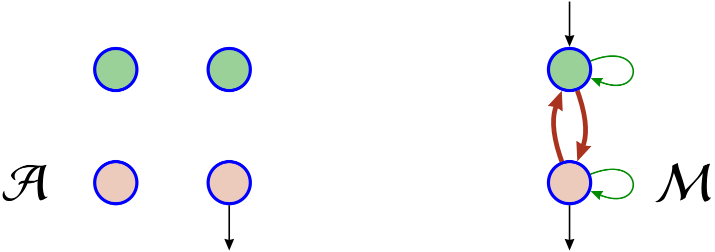


Ultimate equivalence (1) – Definition

Definition

$\mathcal{A}$: a complete DFA.
s, t: states of $\mathcal{A}$.
m: an integer.

*s and t are m-**ultimately-equivalent** (w.r.t. $\mathcal{A}$) if,*

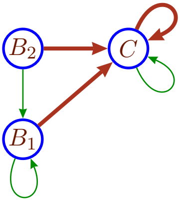
$$\forall \text{ word } u \text{ of length } m, \left[s \xrightarrow{u} s' \text{ and } t \xrightarrow{u} t' \text{ implies } s' = t' \right].$$

Remarks

- s and t are **not** m-ult-equiv

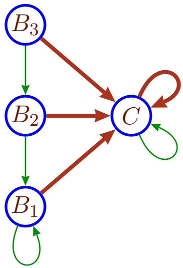
$$\iff \exists \text{ word } u \text{ of length } m, \begin{cases} s \xrightarrow{u} s' \\ t \xrightarrow{u} t' \\ s' \neq t' \end{cases}$$

- s and t are m-ult-equiv $\implies$ s and t are $(m+1)$-ult-equiv.



- B_1 and B_2 are 1-ult-equiv.
- All others pairs are not ult-equiv, as witnessed by the family 0^*.

Ultimate equivalence (2) – Example



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$\forall$ digit d, *s_d: state such that $s \xrightarrow{d} s_d$.*
t_d: state such that $t \xrightarrow{d} t_d$.

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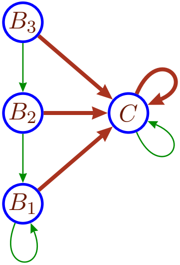
$\iff$ $\forall$ digit d, s_d and t_d are $(m{-}1)$-ult-equiv.



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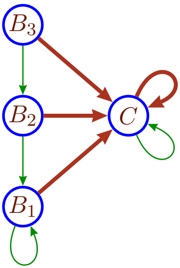

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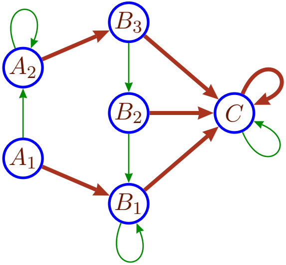

- B_1 and B_2 are 1-ult-equiv.
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- B_3 and B_1 are 2-ult-equiv.
- A_1 and A_2 are 3-ult-equiv.
- All others pairs are not ult-equiv, as witnessed by the family 0^*.

Ultimate equivalence (3) – Computation

$\mathcal{A}$: a DFA.
n: the number of states in $\mathcal{A}$.
b: the size of the alphabet.

Using the automaton product $\mathcal{A} \times \mathcal{A}$, it is known that:

Lemma (folklore)

Ultimate-equivalence w.r.t. $\mathcal{A}$ can be computed in $O(bn^2)$ time.

There exists a better algorithm:

Theorem (Béal-Crochemore, 2007)

Ultimate-equivalence w.r.t. $\mathcal{A}$ can be computed in $O(b\,n\,\log(n))$ time.

Plan

Purely periodic set

Definition

A ***purely periodic*** *set is a set of the form* $R + p\mathbb{N}$ *with*
p*: an integer*
R*: a set of remainders modulo* p

Convention

In the following, p is assumed to be the **smallest period** of $R + p\mathbb{N}$.

The naive automaton $\mathcal{A}_{(p,R)}$ accepting $R + p\mathbb{N}$

b: *the base.*

p: *the period.*

R: *remainder set mod* p.

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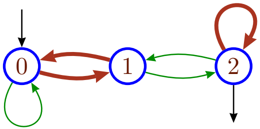

Example 1: $p = 3$, $R = \{2\}$

Definition

$\mathcal{A}_{(p,R)}$:

- *State set:* $\mathbb{Z}/p\mathbb{Z}$
- *Initial state:* 0
- *Transitions:*
  $\forall$ *state* s, $\forall$ *digit* d
  $$s \xrightarrow{d} sb + d$$
- *Final-state set:* R

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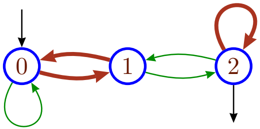

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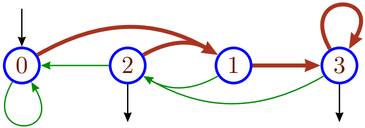

Example 2: $p = 4$, $R = \{2, 3\}$

The naive automaton $\mathcal{A}_{(p,R)}$ accepting $R + p\mathbb{N}$

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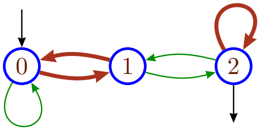

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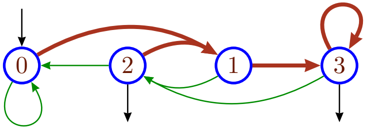

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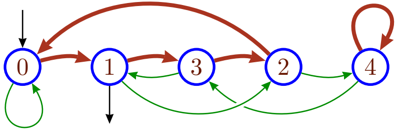


Example 3: $p = 5$, $R = \{1\}$

Property of $\mathcal{A}_{(p,R)}$ in special cases

b: the base.
p: the period.
R: remainder set mod p.

Lemma

p and b are coprime $\implies$ $\mathcal{A}_{(p,R)}$ is a group automaton.

($\forall$ digit d, "reading d" is permutation of the states of $\mathcal{A}_{(p,R)}$)

Lemma

p divides a power of b $\implies$ all states of $\mathcal{A}_{(p,R)}$ are ult-equiv.

$\mathcal{A}_{(p,R)}$ as the product $\mathcal{A}_{(k,?)} \times \mathcal{A}_{(d,?)}$

Notation

b: *the base*

p: *the period*

---

k, d, j: *integers s. t.*

- $p = kd$
- k *coprime with* b
- d *divides* b^j
- k *coprime with* d

Ex.: with $p = 12$,

- $12 = 4 \times 3$
- 4 divides 2^2
- 3 is coprime with 2

Notation

b: *the base*

p: *the period*

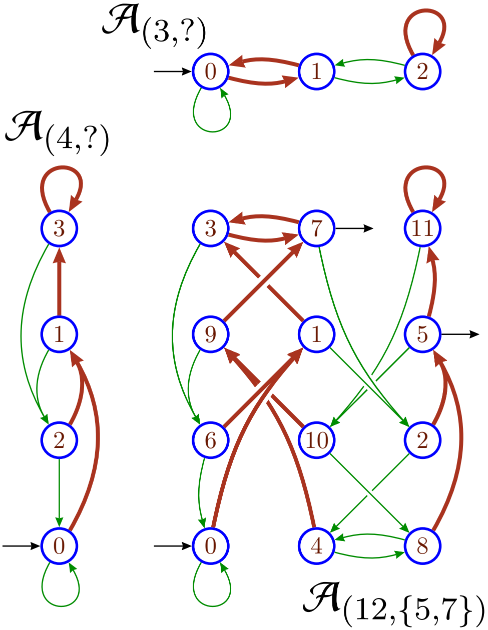
---

k, d, j: *integers s. t.*

- $p = kd$
- k *coprime with* b
- d *divides* b^j
- k *coprime with* d

Ex.: with $p = 12$,

- $12 = 4 \times 3$
- 4 divides 2^2
- 3 is coprime with 2



The "vertical" pseudo-morphism $\mathcal{A}_{(p,R)} \to \mathcal{A}_{(k,?)}$

b: the base
$p = k\,d$: the period
k is coprime with b
d divides b^j

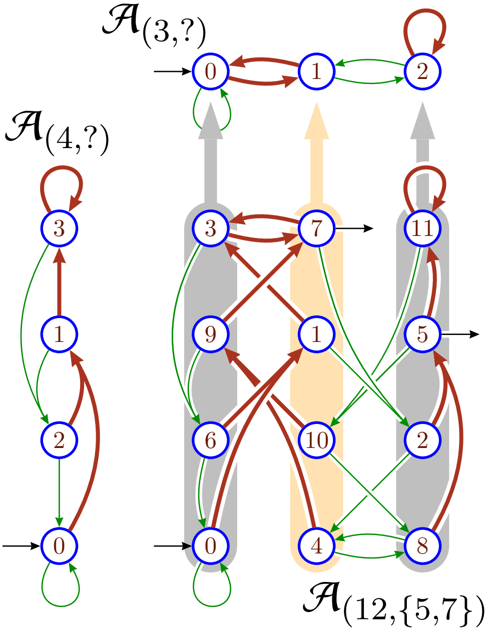
Lemma

$\exists$ a pseudo-morphism
$\psi : \mathcal{A}_{(p,R)} \to \mathcal{A}_{(k,?)}$

Lemma

s, t: states

If $\psi(s) = \psi(t)$, then
s and t are ult-equiv.



b: *the base*

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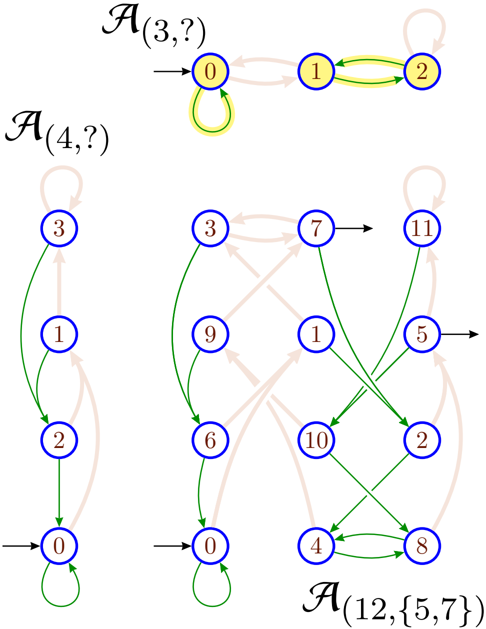
Lemma

- *In* $\mathcal{A}_{(k,?)}$, *all states belong to a 0-circuit;*



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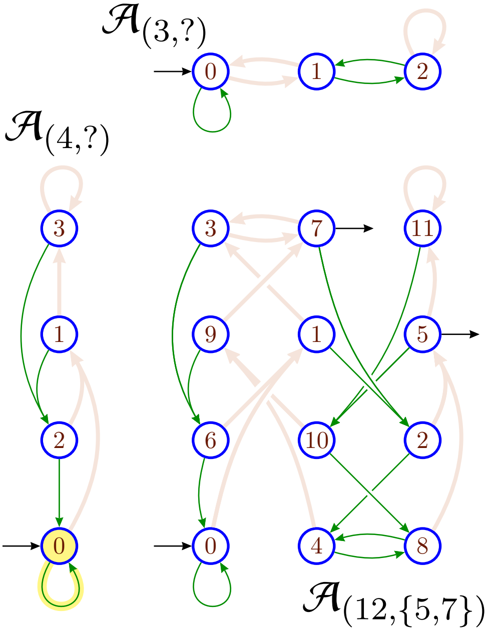
- *In* $\mathcal{A}_{(k,?)}$, *all states belong to a 0-circuit;*
- *In* $\mathcal{A}_{(d,?)}$, *only the initial state is part of a 0-circuit.*

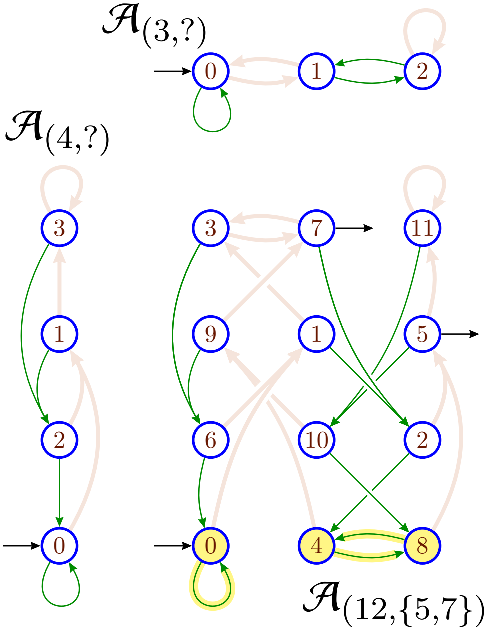


Transition labelled by 0 in $\mathcal{A}_{(p,R)}$

b: *the base*
$p = k\,d$: *the period*
k *is coprime with* b
d *divides* b^j

Lemma

- *In* $\mathcal{A}_{(k,?)}$, *all states belong to a 0-circuit;*
- *In* $\mathcal{A}_{(d,?)}$, *only the initial state is part of a 0-circuit.*
- *In* $\mathcal{A}_{(p,R)}$, k *states are part of a 0-circuit: one by column.*



b: the base
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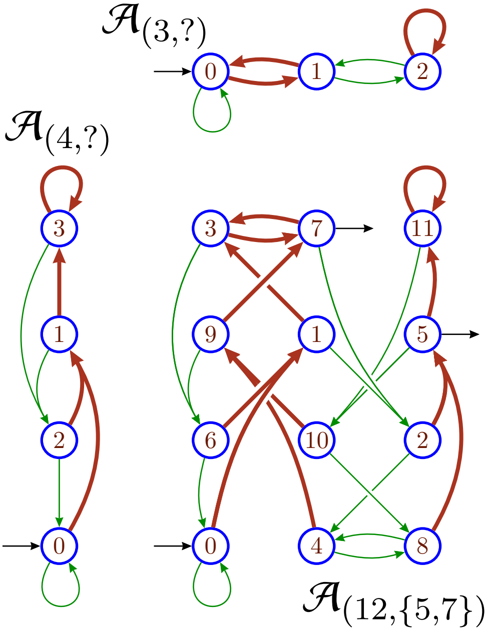
Lemma

States in different columns are never merged by minimisation.

Ex., Ab absurdo: $0 \sim 4$
$\implies$ $0 \sim 4 \sim 8$
$\implies$ $9 \sim 1 \sim 5$
$\implies$ $6 \sim 10 \sim 2$
$\implies$ 4 is a period < 12
$\implies$ Contradiction



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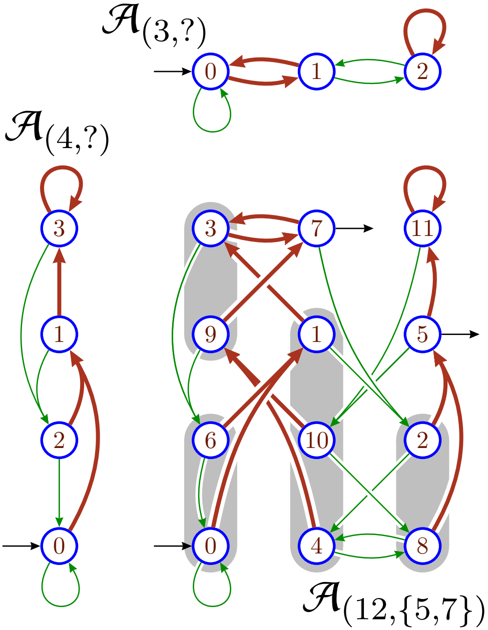

$\mathcal{M}_{(p,R)}$, the minimisation of $\mathcal{A}_{(p,R)}$

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$p = k\,d$: the period
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d divides b^j

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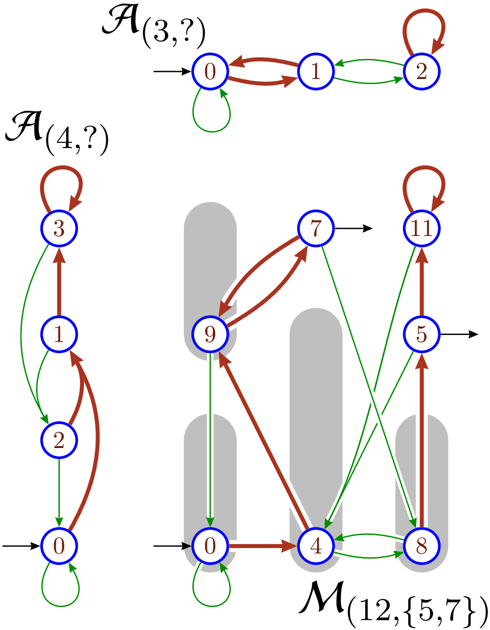


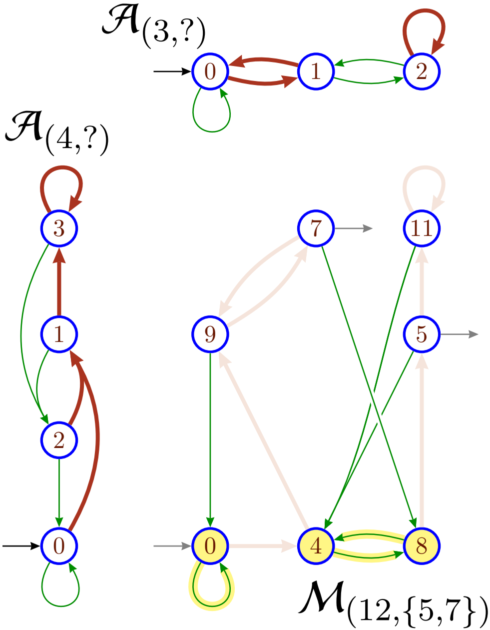
Extracting information from $\mathcal{M}_{(p,R)}$.

b: *the base*
$p = k\,d$: *the period*
k *is coprime with* b
d *divides* b^j

Proposition

$\mathcal{M}_{(p,R)}$: minim. of $\mathcal{A}_{(p,R)}$

- k states of $\mathcal{M}_{(p,R)}$ are part of a 0-circuit



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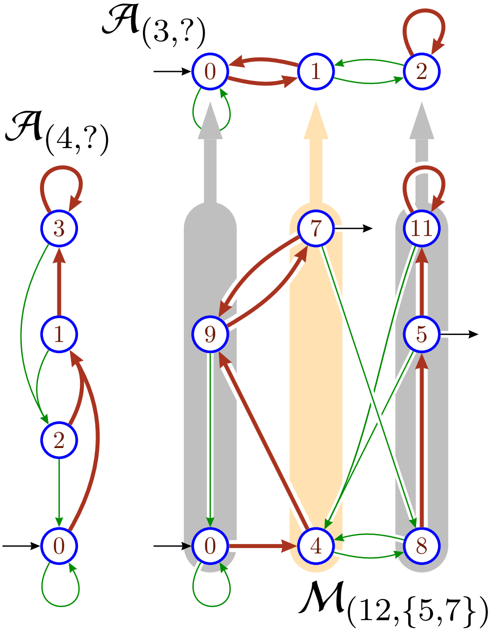
- k states of $\mathcal{M}_{(p,R)}$ are part of a 0-circuit
- $\exists$ a pseudo-morphism $\theta : \mathcal{M}_{(p,R)} \to \mathcal{A}_{(k,?)}$



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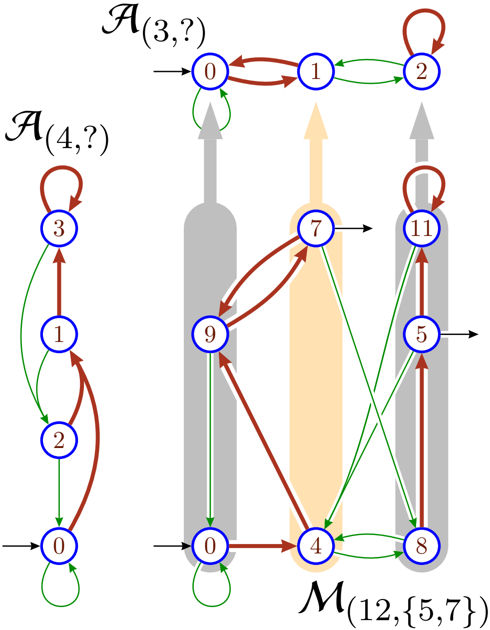

Plan

Theorem

$\mathcal{A}$*: a minimal DFA.*
X*: the b-recognisable set accepted by* $\mathcal{A}$*.*
ℓ*: the total number of states in 0-circuits.*

X *is purely periodic if and only if*

- $\exists$ *a pseudo-morphism* $\varphi : \mathcal{A} \to \mathcal{A}_{(\ell,?)}$*;*
- *states* s, t *such that* $\varphi(s) = \varphi(t)$*, are ultimately equivalent;*
- *the initial state of* $\mathcal{A}$ *bears a 0-loop.*

Forward direction

$\mathcal{A}$: a minimal DFA.
X: the b-recognisable set accepted by $\mathcal{A}$.
ℓ: the total number of states in 0-circuits.

Sketch

- Hypothesis: $\exists p, R, \quad X = R + p\mathbb{N}$

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- Hypothesis: $\exists p, R, \quad X = R + p\mathbb{N}$
- $\mathcal{A}$ is minimal $\quad\Rightarrow\quad$ $\mathcal{A}$ is the minimisation of $\mathcal{A}_{(p,R)}$

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- Prop. of $\mathcal{M}_{(p,R)}$ $\Rightarrow$
  - It holds $k = \ell$
  - Let $\varphi = \theta$
  - states s, t such that $\theta(s) = \theta(t)$ are ultimately-equivalent.

Backward direction

$\mathcal{A}$: a minimal DFA.

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ℓ: the total number of states in 0-circuits.

Sketch

- Hypothesis: $\begin{cases} \exists\varphi, \text{ pseudo-morphism } \mathcal{A} \to \mathcal{A}(\ell, ?) \\ \text{states } s, t \text{ such that } \varphi(s) = \varphi(t) \text{ are ult-equiv.} \end{cases}$

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- Notation: m the maximal bound for ultimate-equivalence.

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  - Let u, u' two words such that $\text{VAL}(u) = \text{VAL}(u')$ $[b^m \ell]$.

Backward direction

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Backward direction

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  - Arithmetic... $\quad \Rightarrow \quad \mathrm{VAL}(v) = \mathrm{VAL}(v')\ [\ell]$

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  - In $\mathcal{A}$, v and v' reach states whose images by φ are equal

Backward direction

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  - In $\mathcal{A}$, v and v' reach states whose images by φ are equal
  - In $\mathcal{A}$, $u = vw$ and $u' = v'w$ reach the same state.

Plan

0 Start from a minimal complete DFA $\mathcal{A}$.

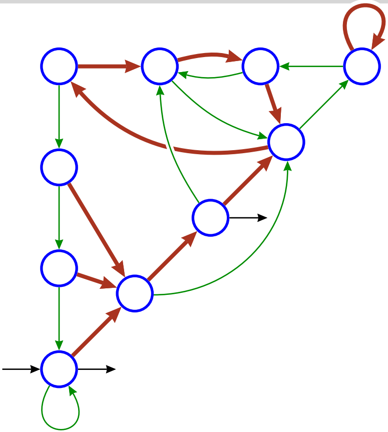
1 Count the number ℓ of states in 0-circuits.

2 Build $\mathcal{A}_{(\ell,?)}$.

3 Compute the pseudo-morphism $\varphi : \mathcal{A} \to \mathcal{A}_{(\ell,?)}$.

4 Check that states s, t such that $\varphi(s) = \varphi(t)$ are ult-equiv.

Execution on an example



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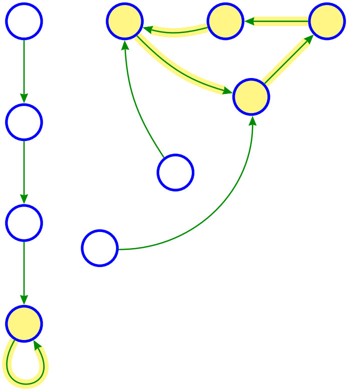

0 Start from a minimal complete DFA $\mathcal{A}$.

1 Count the number ℓ of states in 0-circuits. $\boldsymbol{\ell = 5}$

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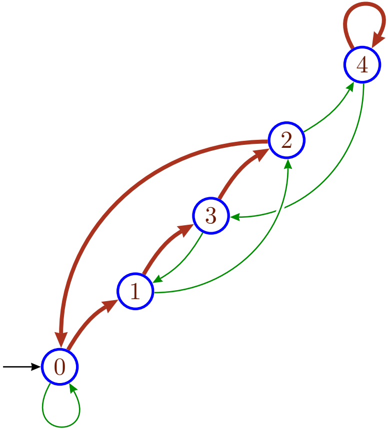

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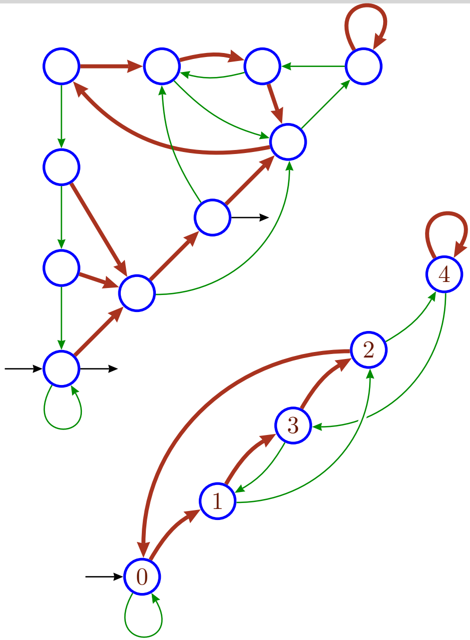

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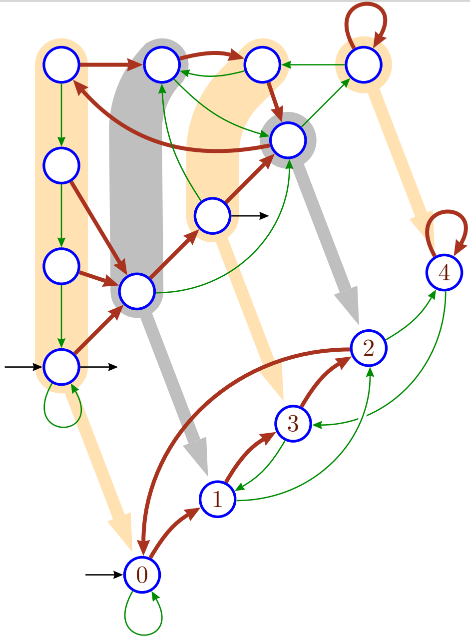

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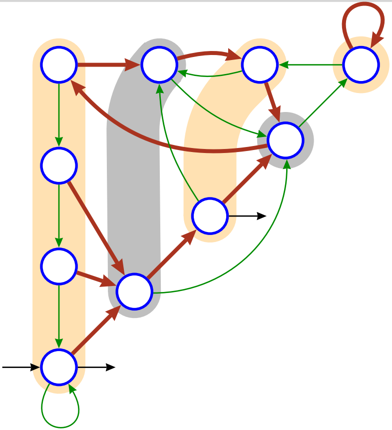

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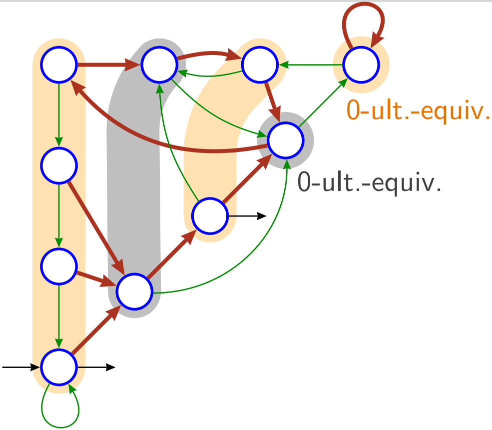

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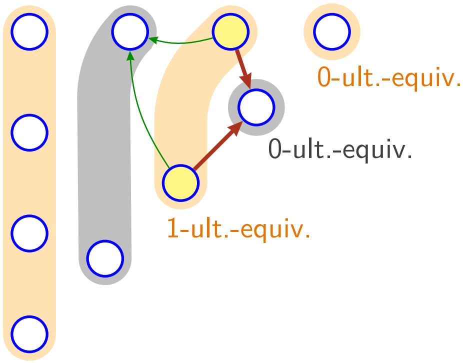

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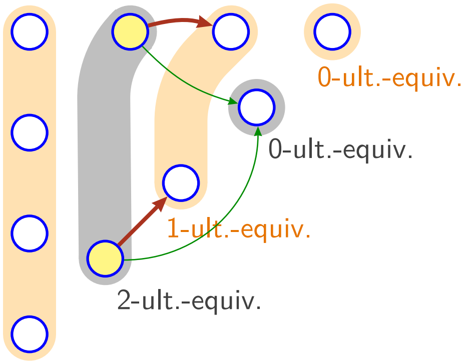

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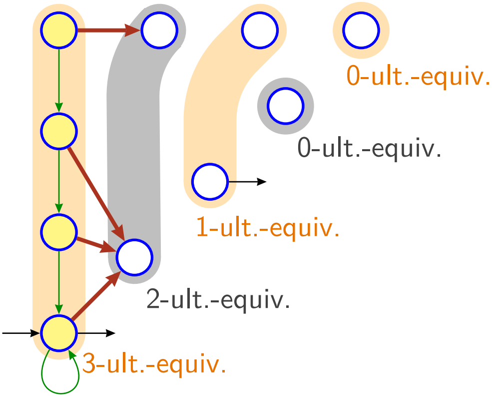

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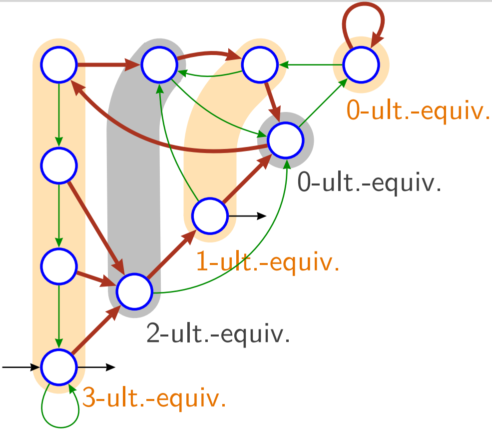

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$$\left(\begin{array}{c} \text{Then, the period is} \\ b^m \times \ell = 2^3 \times 5 = 40 \end{array} \right)$$

Plan

Impurely periodic sets

Definition

A set S is ***impurely periodic*** $\iff$ *S is eventually periodic but not purely periodic*

Impurely periodic sets

Definition

*A set S is **impurely periodic*** $\Longleftrightarrow$ *S is eventually periodic but not purely periodic*

Theorem

$\mathcal{A}$: a minimal DFA.
S: the b-recognisable set accepted by $\mathcal{A}$.
*ℓ: the total number of states in 0-circuits **minus one**.*

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*S is **im**purely periodic if and only if*

- *$\exists$ a pseudo-morphism $\varphi : \mathcal{A} \to \mathcal{A}_{(\ell,?)}$;*
- *every **non-initial** states s, s' such that $\varphi(s) = \varphi(s')$, are ultimately equivalent;*
- *the initial state of $\mathcal{A}$ bears a 0-loop **and has no other incoming transitions**.*

Conclusion

Eventually periodic sets are either purely or impurely periodic, hence:

Theorem

PERIODICITY *is decidable in* $O(b\,n\,\mathit{log}(n))$ *time*

(where n *is the state-set cardinal.)*

Future work

- *Extension to multi-dimensional sets.*
- *Extension to non-standard numeration systems.*