The Successor Function for Minimal Words in Rational Base Numeration Systems

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joint work with Shigeki Akiyama and Jacques Sakarovitch

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Plan

Integer base (1)

- Base: $p \geq 2$
- Alphabet: $A_p = \{0, 1, \ldots, p-1\}$

Eg., in base 3, numbers are represented using the digits 0, 1 and 2.

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Evaluation (Words $\rightarrow$ Numbers)

$\forall a_n \cdots a_1 a_0 \in A_p^*$

$$\pi_p(a_n \cdots a_1 a_0) = a_n\, p^n + \cdots + a_1\, p^1 + a_0 = \sum_{i=0}^{n} a_i p^i$$

Eg., in base 3, $\pi_3(0121) = 0 \times 27 + 1 \times 9 + 2 \times 3 + 1 \times 1 = 16$

Integer base (2)

Representation (Numbers $\to$ Words)

- Right-to-left : Euclidean division algorithm
  - $\forall n > 0 \quad \langle n \rangle_p = \langle n' \rangle_p \, a,$ where (n', a) is the Euclidean division of n by p.
  - $\langle 0 \rangle_p = \varepsilon$
- Left-to-right: greedy algorithm...

Eg., in base 3,

$$
\begin{aligned}
\langle 14 \rangle_3 &= \langle 4 \rangle_3 \, 2, & &\text{since } 14 = 4 \times 3 + 2 \\
&= \langle 1 \rangle_3 \, 1 \, 2, & &\text{since } 4 = 1 \times 3 + 1 \\
&= \langle 0 \rangle_3 \, 1 \, 1 \, 2, & &\text{since } 1 = 0 \times 3 + 1 \\
&= \varepsilon \, 1 \, 1 \, 2 \\
&= 1 \, 1 \, 2
\end{aligned}
$$

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Eg., in base 3,

$$\begin{aligned} \langle 14 \rangle_3 &= \langle 4 \rangle_3 \, 2, && \text{since } 14 = 4 \times 3 + 2 \\ &= \langle 1 \rangle_3 \, 1\,2, && \text{since } 4 = 1 \times 3 + 1 \\ &= \langle 0 \rangle_3 \, 1\,1\,2, && \text{since } 1 = 0 \times 3 + 1 \\ &= \varepsilon\,1\,1\,2 \\ &= 1\,1\,2 \end{aligned}$$

- $\pi_p(A_p{}^*) = \mathbb{N}$
- $\langle \mathbb{N} \rangle_p = (A_p \setminus \{0\}) \, A_p{}^*$

Rational Base [Akiyama Frougny Sakarovitch 2008]

- Base: $\frac{p}{q}$ where
  - p and q are coprime integers
  - $p > q > 1$
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Eg., in base $\frac{3}{2}$, numbers are represented using the digits 0, 1 and 2.

Evaluation (Words → Numbers)

$\forall a_n \cdots a_1 a_0 \in A_p^*$

$$\pi_p(a_n \cdots a_1 a_0) = \frac{a_n}{q}\left(\frac{p}{q}\right)^n + \cdots + \frac{a_1}{q}\left(\frac{p}{q}\right)^1 + \frac{a_0}{q} = \sum_{i=0}^{n} \frac{a_i}{q}\left(\frac{p}{q}\right)^i$$

Eg., in base $\frac{3}{2}$, $\pi_{\frac{3}{2}}(0212) = \frac{0}{2} \times \frac{27}{8} + \frac{2}{2} \times \frac{9}{4} + \frac{1}{2} \times \frac{3}{2} + \frac{2}{2} \times 1 = 4$

$\pi_{\frac{3}{2}}(1) = \frac{1}{2}$

Representation (Numbers $\rightarrow$ Words)

- Right-to-left: Modified Euclidean Division Algorithm (MED)
  - $\forall n > 0 \quad \langle n \rangle_{\frac{p}{q}} = \langle n' \rangle_{\frac{p}{q}} \, a,$ where (n', a) is the Euclidean division of $(q\,n)$ by p.
  - $\langle 0 \rangle_{\frac{p}{q}} = \varepsilon$

Eg., in base $\frac{3}{2}$,
$$\begin{aligned}
\langle 8 \rangle_{\frac{3}{2}} &= \langle 5 \rangle_{\frac{3}{2}}\, 1 , && \text{since } (8 \times 2) = 5 \times 3 + 1 \\
&= \langle 3 \rangle_{\frac{3}{2}}\, 1\,1 , && \text{since } (5 \times 2) = 3 \times 3 + 1 \\
&= \langle 2 \rangle_{\frac{3}{2}}\, 0\,1\,1 , && \text{since } (3 \times 2) = 2 \times 3 + 0 \\
&= \langle 1 \rangle_{\frac{3}{2}}\, 1\,0\,1\,1 , && \text{since } (2 \times 2) = 1 \times 3 + 1 \\
&= 2\,1\,0\,1\,1 , && \text{since } (1 \times 2) = 0 \times 3 + 2
\end{aligned}$$

Rational Base (2)

Representation (Numbers $\rightarrow$ Words)

- Right-to-left: Modified Euclidean Division Algorithm (MED)
  - $\forall n > 0 \quad \langle n \rangle_{\frac{p}{q}} = \langle n' \rangle_{\frac{p}{q}}\, a,$ where (n', a) is the Euclidean division of $(q\, n)$ by p.
  - $\langle 0 \rangle_{\frac{p}{q}} = \varepsilon$
- Left-to-right: **no such algorithm !**

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Insights on counting in rational base

Addition in base 3

	1	1	1
+		2	2
	1	3	3
		+1	-3
	1	4	0
	+1	-3	
	2	1	0

Carry Rule: +1 -3

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Addition in base $\frac{3}{2}$

Carry Rule: +2 -3

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	1	1	1
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	1	3	3
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	1	4	0
	+1	-3	
	2	1	0

Carry Rule: +1 -3

Addition in base $\frac{3}{2}$

	1	1	1
+		2	2
	1	3	3
		+2	-3
	1	5	0
	+2	-3	
	3	2	1
+2	-3		
2	0	2	1

Carry Rule: +2 -3

Insights on counting in rational base (2)

Knowing that

- carry rule for base $\frac{p}{q}$: $+q$ $-p$;
- $\langle 1 \rangle_{\frac{p}{q}}$ is the digit 'q'.

We may compute the representation of any integers:

Eg., computation of $\langle 4 \rangle_{\frac{3}{2}} = \langle 1+1+1+1 \rangle_{\frac{3}{2}}$

$$8 \quad (= q + q + q + q)$$

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We indeed just redefined the MED algorithm !

$L_{\frac{p}{q}}$: the set of the representations of integers in base $\frac{p}{q}$.

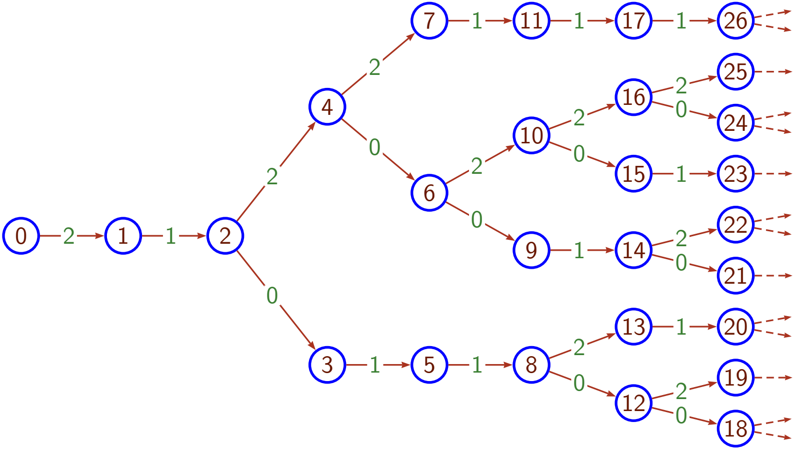

Figure: The language $L_{\frac{3}{2}}$ represented as a tree

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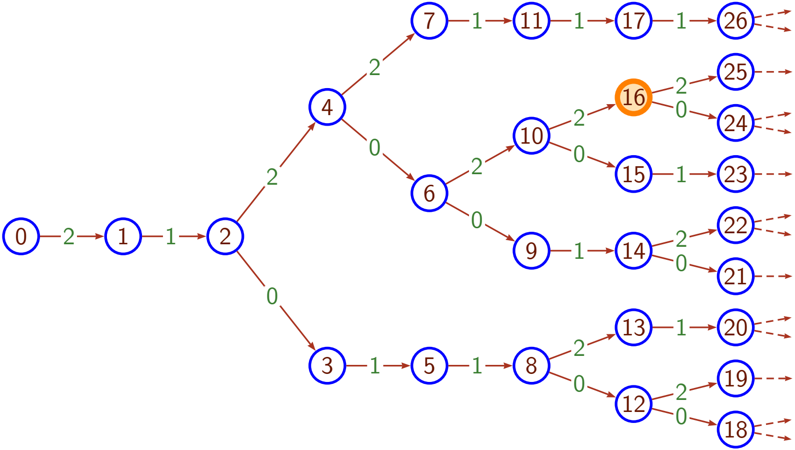

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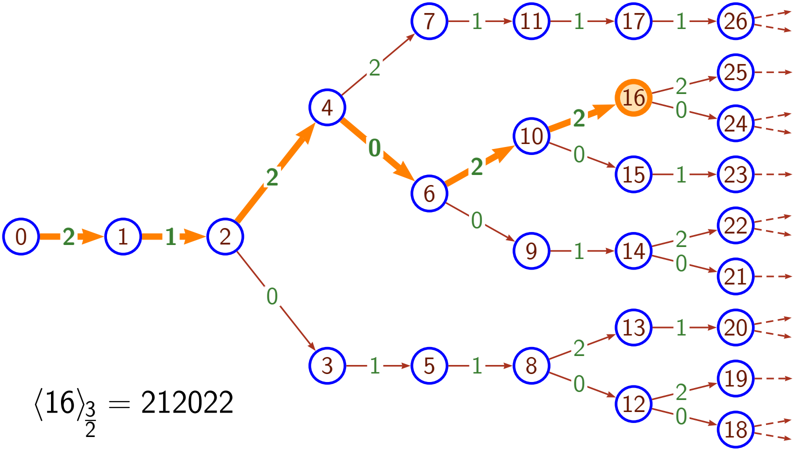

$\langle 16 \rangle_{\frac{3}{2}} = 212022$

Figure: The language $L_{\frac{3}{2}}$ represented as a tree

The language $L_{\frac{p}{q}}$ and the automaton $\mathcal{T}_{\frac{p}{q}}$

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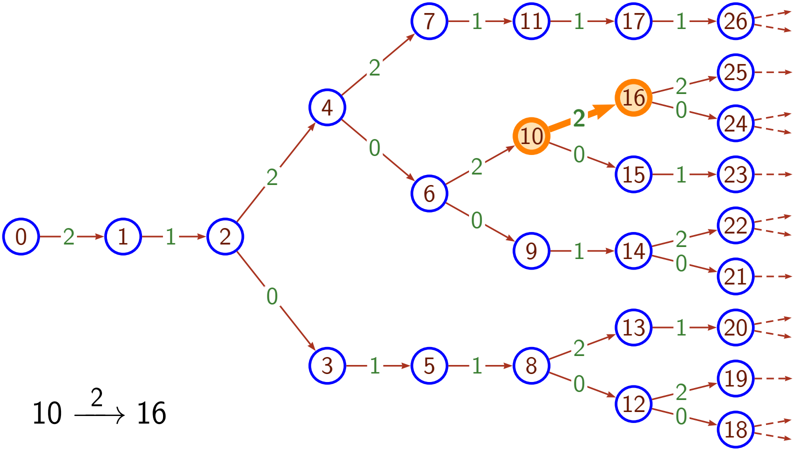

$10 \xrightarrow{2} 16$

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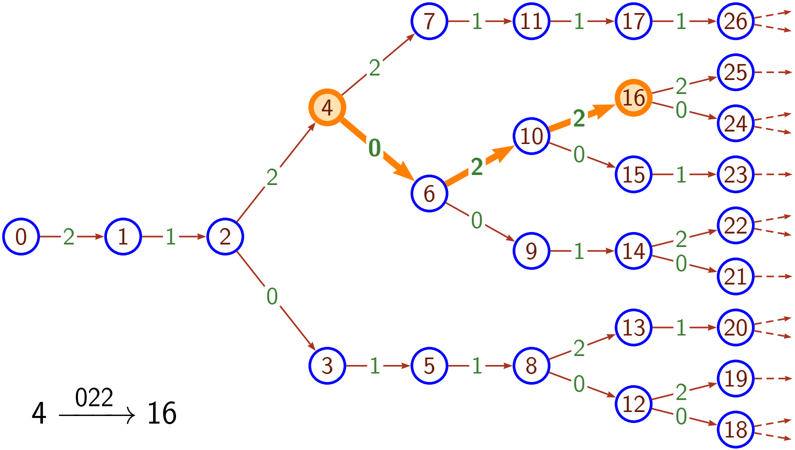


$$4 \xrightarrow{022} 16$$

Figure: The language $L_{\frac{3}{2}}$ represented as a tree

$\mathcal{T}_{\frac{p}{q}}$: the **infinite** automaton accepting $0^* L_{\frac{p}{q}}$.

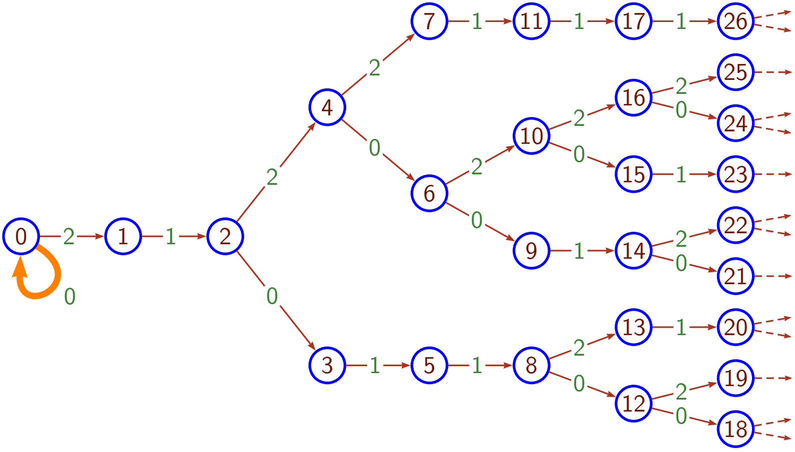


Figure: The infinite automaton $\mathcal{T}_{\frac{3}{2}}$

Formal definition of the automaton $\mathcal{T}_{\frac{p}{q}}$

$$\forall a \in A_p \quad \forall n, m \in \mathbb{N} \qquad n \xrightarrow{a} m \iff n\,p + a \;=\; q\,m$$

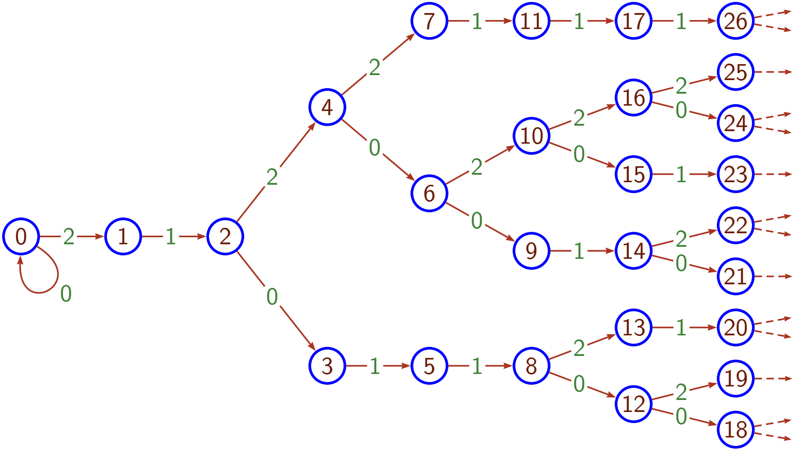
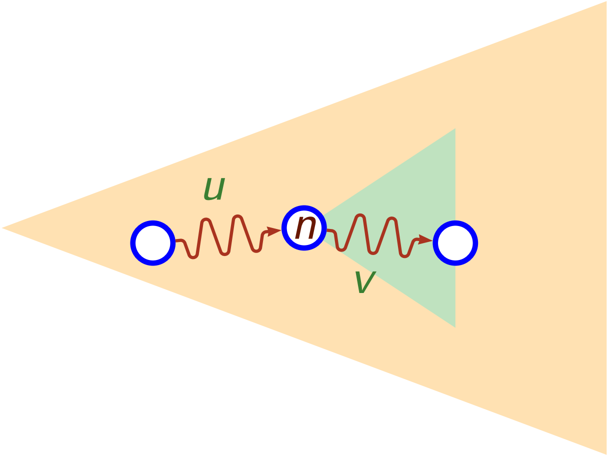


Figure: The infinite automaton $\mathcal{T}_{\frac{3}{2}}$

Past and future within $\mathcal{T}_{\frac{p}{q}}$ (1)



- Past is characterised by congruency modulo powers of p
- Future -//- of q

Past and future within $\mathcal{T}_{\frac{p}{q}}$ (2)

u: a word of length k
n, n': two integers

Future Lemma [AFS'08]

Any two of the following implies the third

(i) $n \xrightarrow{u} \cdot$
(ii) $n' \xrightarrow{u} \cdot$
(iii) $n \equiv n' \; [q^k]$

Past Lemma [AFS'08]

Any two of the following implies the third

(i) $\cdot \xrightarrow{u} n$
(ii) $\cdot \xrightarrow{u} n'$
(iii) $n \equiv n' \; [p^k]$

Hence, $L_{\frac{p}{q}}$ is a complex language

Theorem [AFS'08]

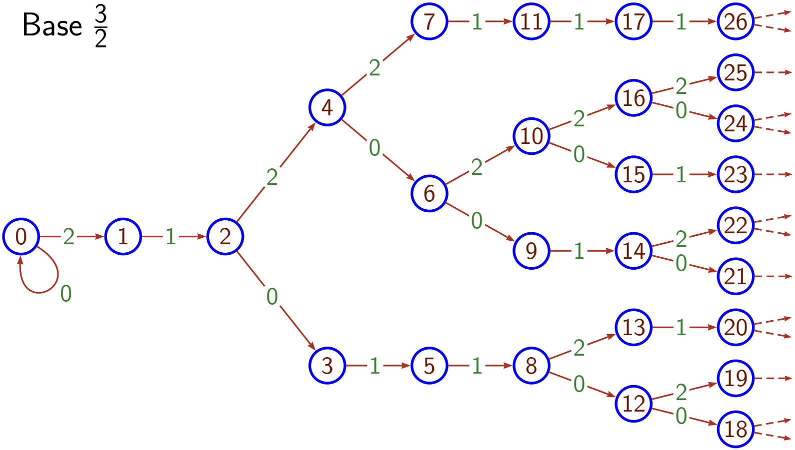
$L_{\frac{p}{q}}$ is not a regular language, nor even a context-free language.

Theorem [AFS'08]

w: an infinite whose every prefix belongs to $L_{\frac{p}{q}}$.
Then, w is aperiodic.

Hence, $\mathcal{T}_{\frac{3}{2}}$ possess a strong “transversal” regularity

Base $\frac{3}{2}$



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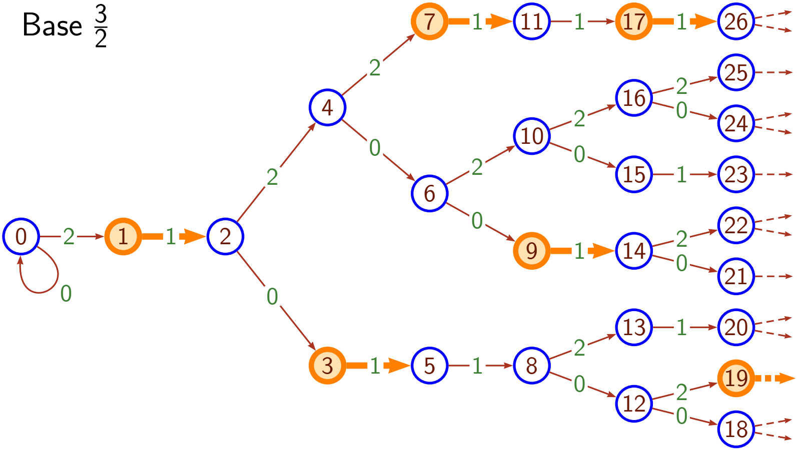
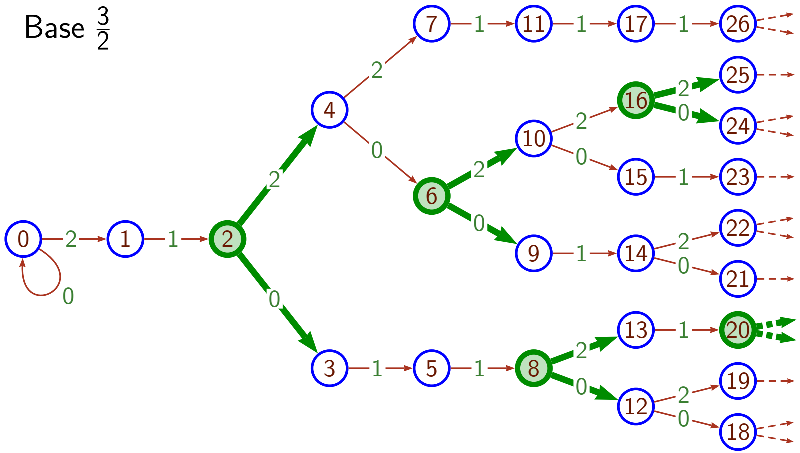


Every odd node has one outgoing arc labelled by 1.

Every even node has two outgoing arcs labelled by 0 and 2.

Hence, $\mathcal{T}_{\frac{3}{2}}$ possess a strong “transversal” regularity



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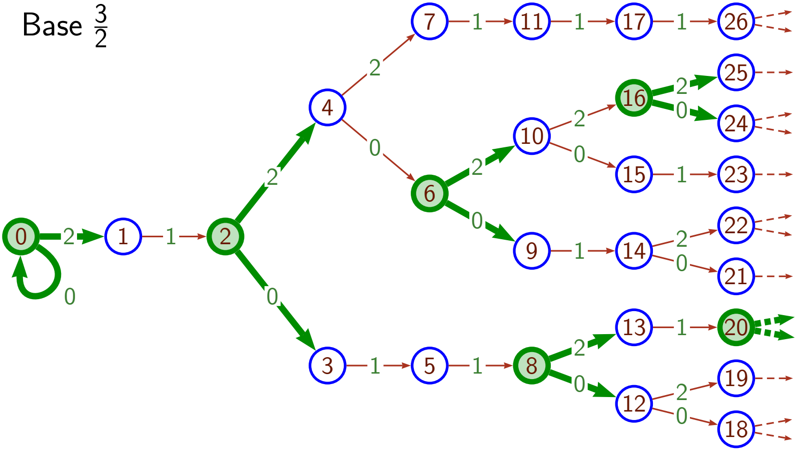

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Building $\mathcal{T}_{\frac{3}{2}}$ periodically with $2, 1, 2, 1, \ldots$

Base $\frac{3}{2}$

Every odd node has one outgoing arc labelled by 1.

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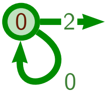

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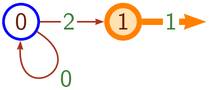

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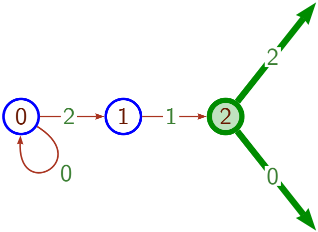

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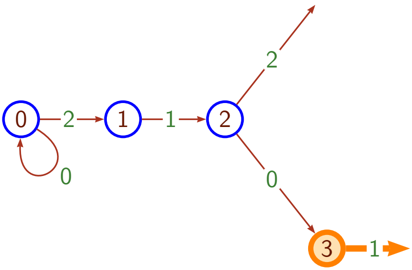

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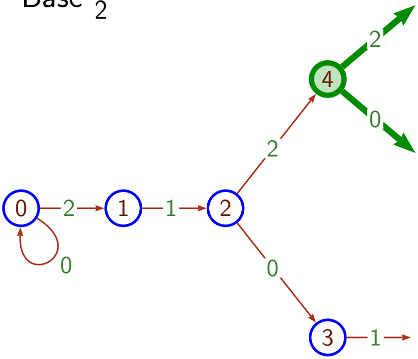

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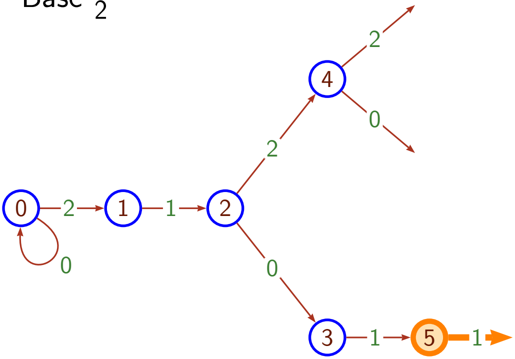

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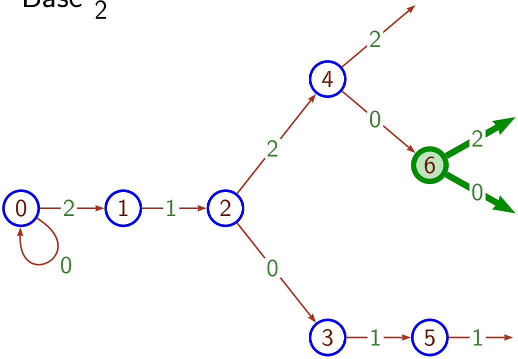

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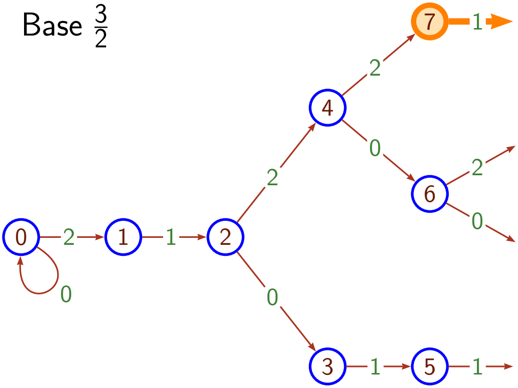

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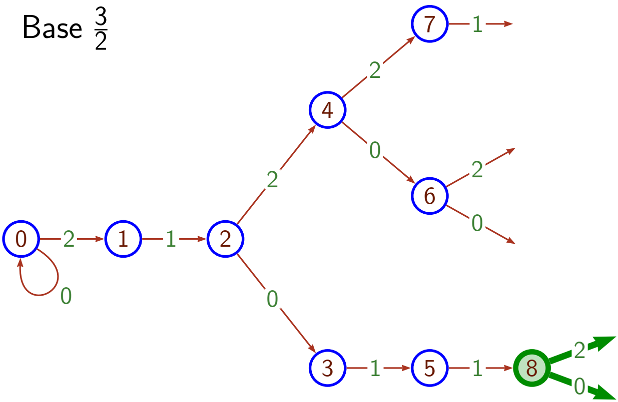

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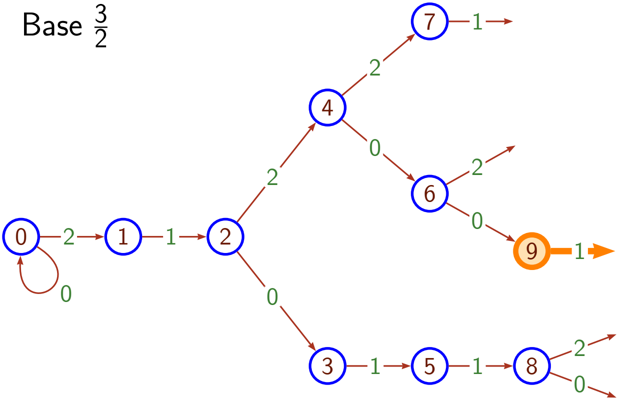

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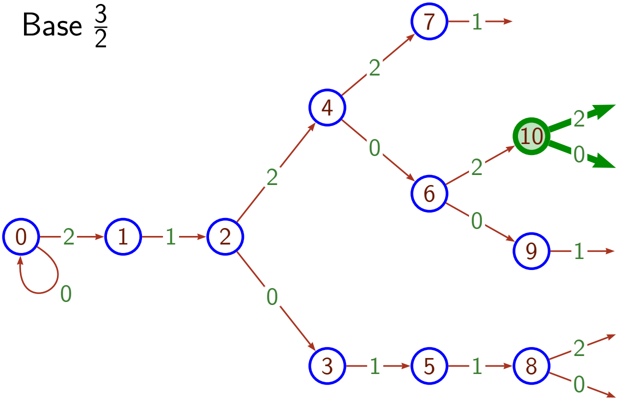

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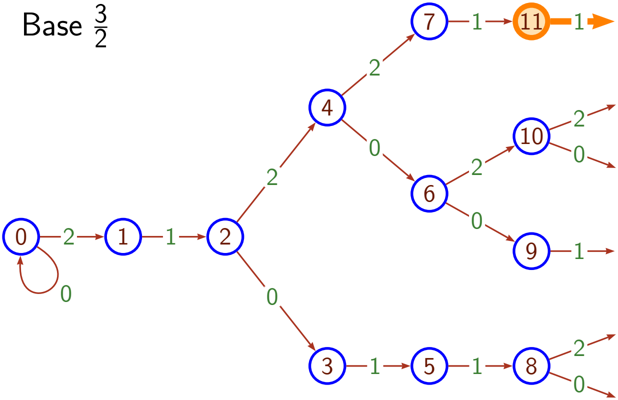

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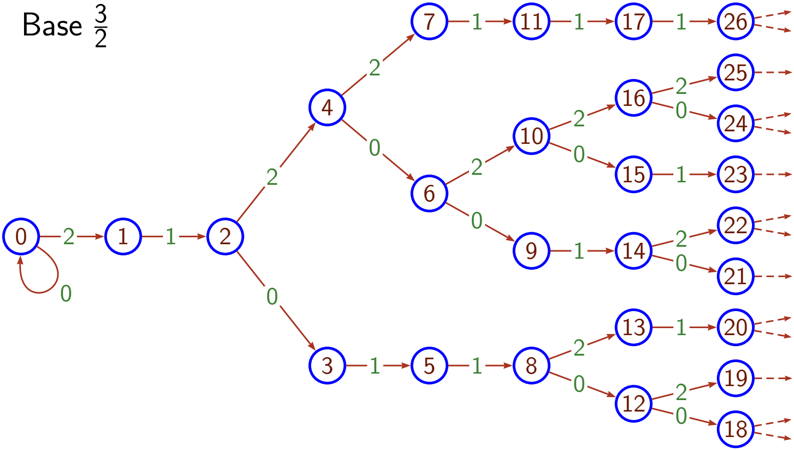

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Plan

Infinite words

Topology on infinite words

- Distance between two words is 2^{-i} if their longest common prefix has length i.
- A finite word u is considered as $u\#^\omega$ ($\#$ is a special letter).
- The topological closure $\mathtt{adh}(L)$ of $L \in (A^* \cup A^\omega)$ is then:

$$\mathtt{adh}(L) = \left\{ w \in A^\omega \;\middle|\; \forall i \in \mathbb{N},\quad \begin{array}{l} \exists w' \in L,\quad w \text{ and } w' \text{ have} \\ \text{the same prefix of length } i \end{array} \right\}$$

Definition

$$W_{\frac{p}{q}} = \mathtt{adh}(0^* L_{\frac{p}{q}})$$

or, equivalently,

$$W_{\frac{p}{q}} = \{\text{infinite words whose run may continue forever in } \mathcal{T}_{\frac{p}{q}}\}$$

Definition

$$\rho_{\frac{p}{q}} : \quad A_p^{\omega} \longrightarrow \mathbb{R}$$

$$a_1 a_2 \cdots a_k \cdots \longmapsto \sum_{k \geq 1} \frac{a_k}{q} \left(\frac{p}{q}\right)^{-k}$$

Ex: $\rho_{\frac{p}{q}}(1^{\omega}) = 1\,\frac{1}{2}\left(\frac{3}{2}\right)^{-1} + 1\,\frac{1}{2}\left(\frac{3}{2}\right)^{-2} + \cdots = \frac{1}{2}\,\frac{\frac{2}{3}}{1-\frac{2}{3}}$

Real-evaluation (*ie.* after the decimal point)

Definition

$$\begin{array}{rccc} \rho_{\frac{p}{q}} : & A_p^{\omega} & \longrightarrow & \mathbb{R} \\ & a_1 a_2 \cdots a_k \cdots & \longmapsto & \displaystyle\sum_{k \geq 1} \frac{a_k}{q} \left(\frac{p}{q}\right)^{-k} \end{array}$$

Ex: $\rho_{\frac{p}{q}}(1^{\omega}) \;=\; 1\frac{1}{2}\left(\frac{3}{2}\right)^{-1} + 1\frac{1}{2}\left(\frac{3}{2}\right)^{-2} + \cdots \;=\; \frac{1}{2}\,\frac{\frac{2}{3}}{1-\frac{2}{3}}$

Convention

For real-evaluation, any finite word is assumed to end with 0^{ω}.

Ex: $\rho_{\frac{p}{q}}(21) = 2\frac{1}{2}\left(\frac{3}{2}\right)^{-1} + 1\frac{1}{2}\left(\frac{3}{2}\right)^{-2} = 0.888\cdots$

Fractal view of $W_{\frac{p}{q}}$

Reminder

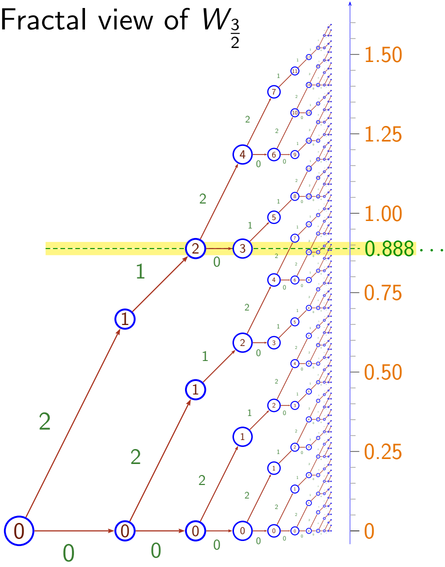
$$\rho_{\frac{p}{q}}(a_1 a_2 \cdots a_k \cdots) = \sum_{k \geq 1} \frac{a_k}{q} \left(\frac{p}{q}\right)^{-k}$$

$$\rho_{\frac{p}{q}}(a_1 a_2 \cdots a_k) = \rho_{\frac{p}{q}}(a_1 a_2 \cdots a_k 0^{\omega})$$

Eg: $\rho_{\frac{3}{2}}(21) = 2\frac{1}{2}\frac{2}{3} + 1\frac{1}{2}\frac{4}{9} = 0.888\cdots$

$\rho_{\frac{3}{2}}(210) = \rho_{\frac{p}{q}}(21) = 0.888\cdots$

Fractal view of $W_{\frac{3}{2}}$



A few infinitary properties

Theorem [AFS'08]

$\rho_{\frac{p}{q}}$ is a continuous and increasing function $(W_{\frac{p}{q}}, \leq_{\text{rad}}) \to (\mathbb{R}, \leq)$.

NB: it is not true for $(A_p^*, \leq_{\text{rad}}) \to (\mathbb{R}, \leq)$.

Theorem [AFS'08]

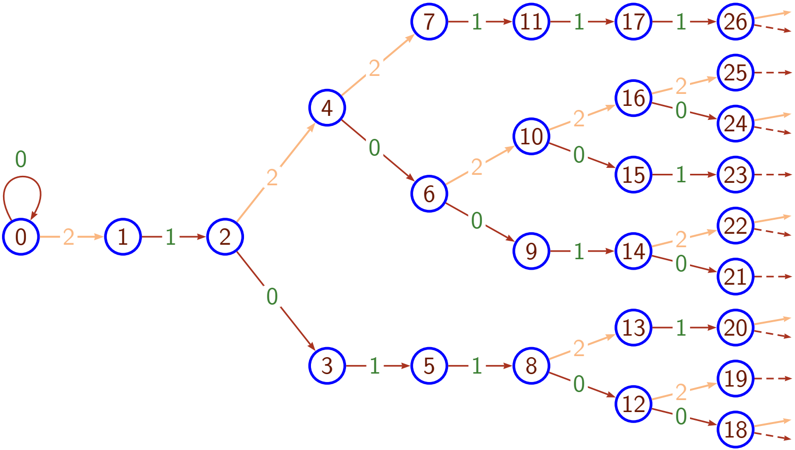
$\rho_{\frac{p}{q}}(W_{\frac{p}{q}})$ is an interval.

Corollary

Real numbers may be represented in rational base numeration systems.

Minimal words – Definition

w_n^-: the (infinite) word starting from n taking the lowest branch.



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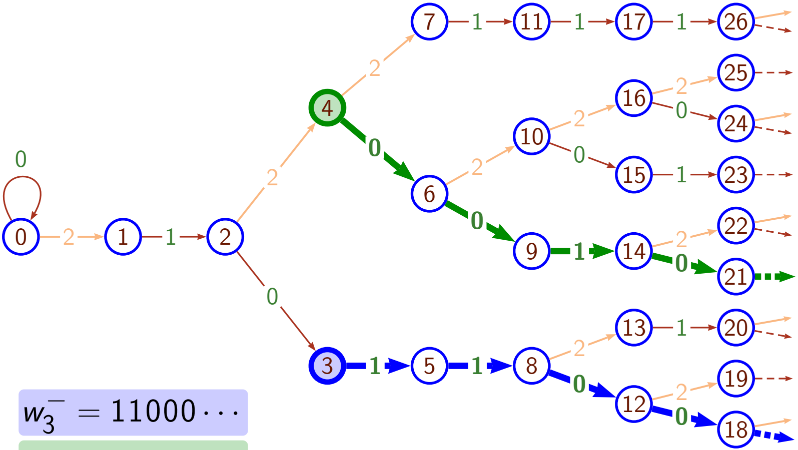
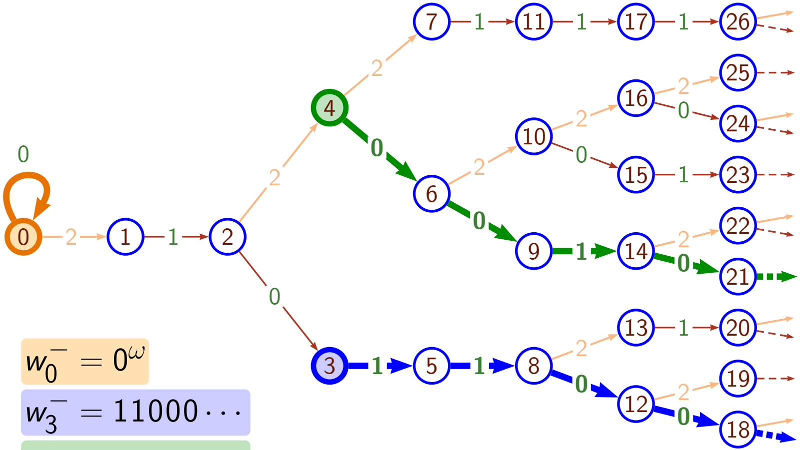


$w_3^- = 11000\cdots$

$w_4^- = 00101\cdots$

w_n^-: the (infinite) word starting from n taking the lowest branch.



$w_0^- = 0^\omega$

$w_3^- = 11000\cdots$

$w_4^- = 00101\cdots$

Minimal words – Properties

Given an integer n, the minimal word w_n^- is

- over the alphabet $\{0, \ldots, (q-1)\} = A_q$
- the unique word over A_q readable from n

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Reformulation of the 'Future Lemma'

$$w_n^- \text{ and } w_m^- \text{ have the same prefix of length } k.$$

$$\Updownarrow$$

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Reformulation of the 'Future Lemma'

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$$\Updownarrow$$
$$n \equiv m \; [q^k]$$

Corollary

- Minimal words are pairwise distinct.
- Every minimal word (but w_0^-) is aperiodic.

Minimal words – Set properties

$\Omega_{\frac{p}{q}}$: the set of minimal words.

NB: $\Omega_{\frac{p}{q}}$ is incomparable with $W_{\frac{p}{q}}$

Properties

- The topological closure of $\Omega_{\frac{p}{q}}$ is A_q^* whole.
- $\Omega_{\frac{p}{q}}$ is countable
- The interior of $\Omega_{\frac{p}{q}}$ is *empty*.

Plan

Successor function

$$\begin{array}{rccc} \xi: & A_q^{\omega} & \longrightarrow & A_q^{\omega} \\ & w_n^- & \longmapsto & w_{n+1}^- \end{array}$$

$$\xi : \begin{array}{ccc} A_q^\omega & \longrightarrow & A_q^\omega \\ w_n^- & \longmapsto & w_{n+1}^- \end{array}$$

Why study this function?

- Could have been simple :
  - $w_n^- = a\, w_{n+p}^-$ for some digit a and integer p (or, equivalently $\xi^p(w_n^-)$ is the shifted of w_n^-).
  - It is "letter-to-letter" (cf. next slide).
- Iterating it browse through $L_{\frac{p}{q}}$.

The successor transducer

Lemma

The successor function ξ is **letter-to-letter**:

$$w \text{ and } w' \text{ have a common prefix of length } i$$
$$\Updownarrow$$
$$\xi(w) \text{ and } \xi(w') \text{ have a common prefix of length i}$$

Proof.

1. Let k, j be such that $w = w_k^-$ and $w' = w_j^-$.
2. By Def. of ξ, $\xi(w) = w_{k+1}^-$ and $\xi(w') = w_{j+1}^-$.
3. w_k^- and w_j^- have a common prefix of length i

   $\iff$ $k \equiv j \; [q^i]$ (From future Lemma)

   $\iff$ $k+1 \equiv j+1 \; [q^i]$

   $\iff$ w_{k+1}^- and w_{j+1}^- have a common prefix of length i

   (From future Lemma again)

The successor transducer

Lemma

The successor function ξ is **letter-to-letter**:

$$w \text{ and } w' \text{ have a common prefix of length } i$$
$$\Updownarrow$$
$$\xi(w) \text{ and } \xi(w') \text{ have a common prefix of length i}$$

$\Rightarrow$ Reading the i-th letter of w_k^- allows to output the i-th letter of w_{k+1}^-.

$\Rightarrow$ The successor function ξ is realised by an infinite-state, letter-to-letter and sequential transducer: $\mathcal{D}_{\frac{p}{q}}$.

Example 1: the remarkable case of base $\frac{3}{2}$

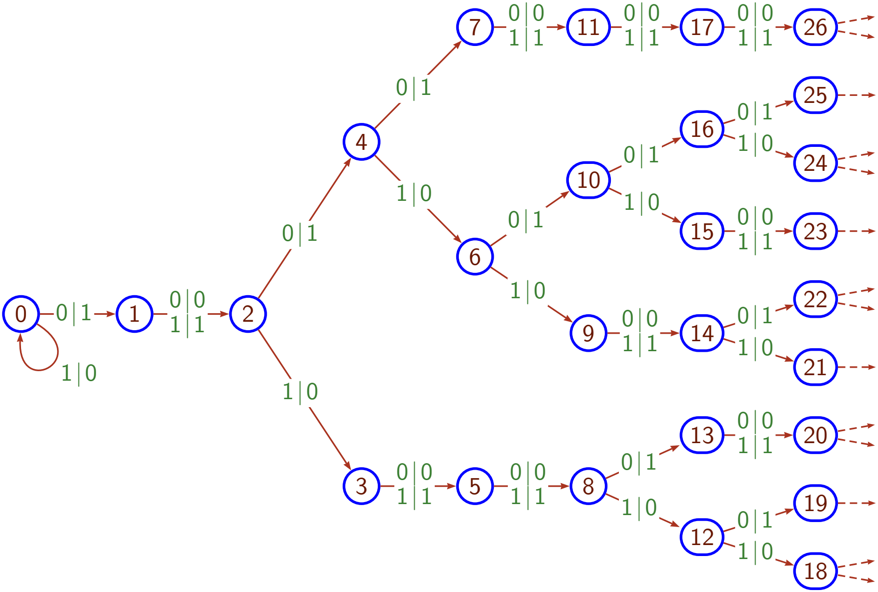

Figure: $\mathcal{D}_{\frac{3}{2}}$, the infinite transducer realising ξ in base $\frac{3}{2}$

Example 1: the remarkable case of base $\frac{3}{2}$

$w_3^- = 1\,1\,0\,0\,0\,1\,1\,0\,1\,\cdots$

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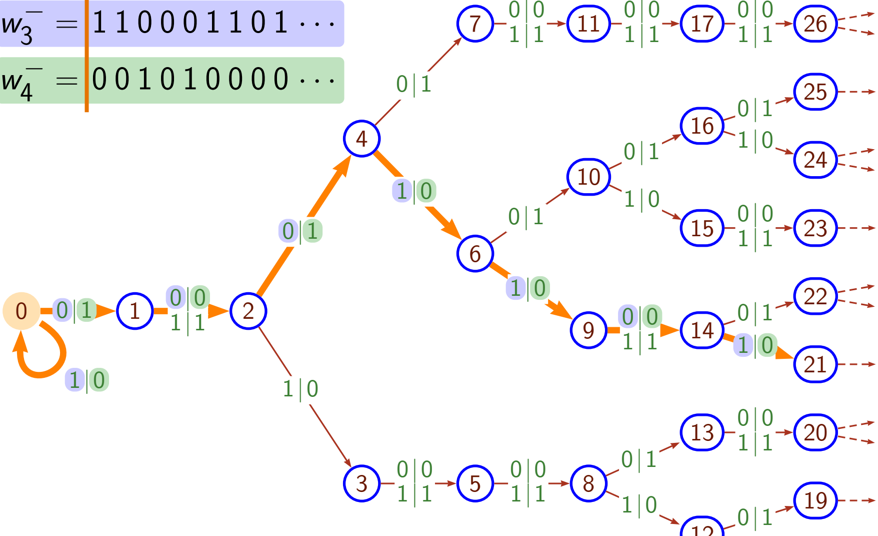

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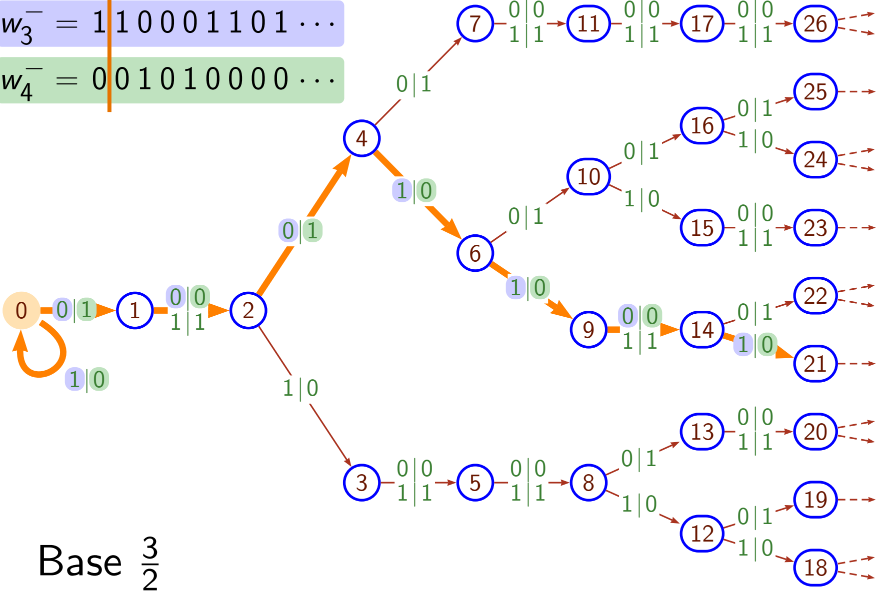

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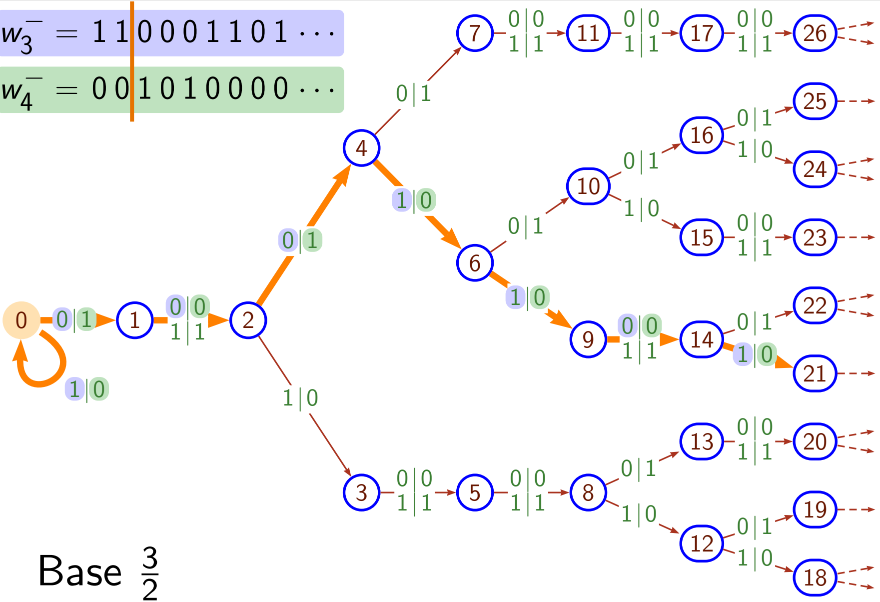

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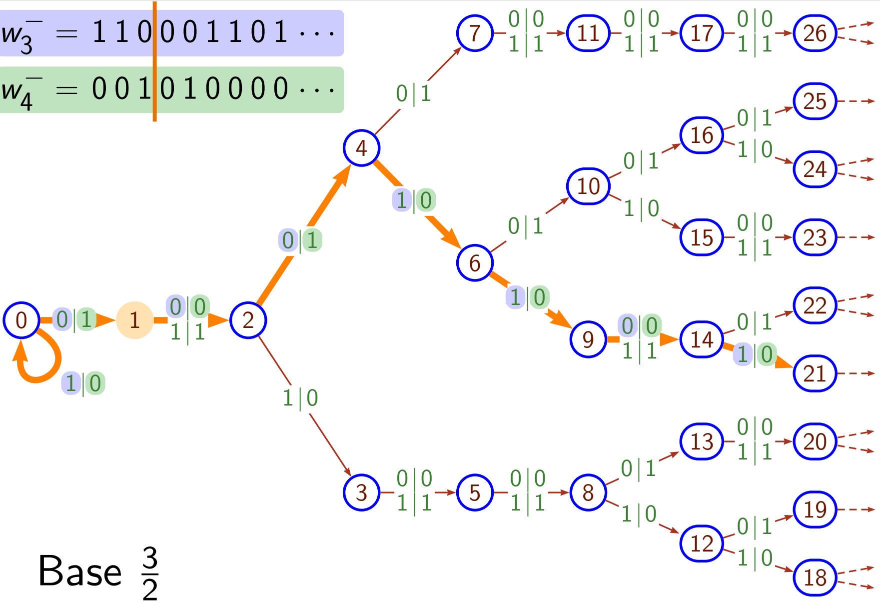

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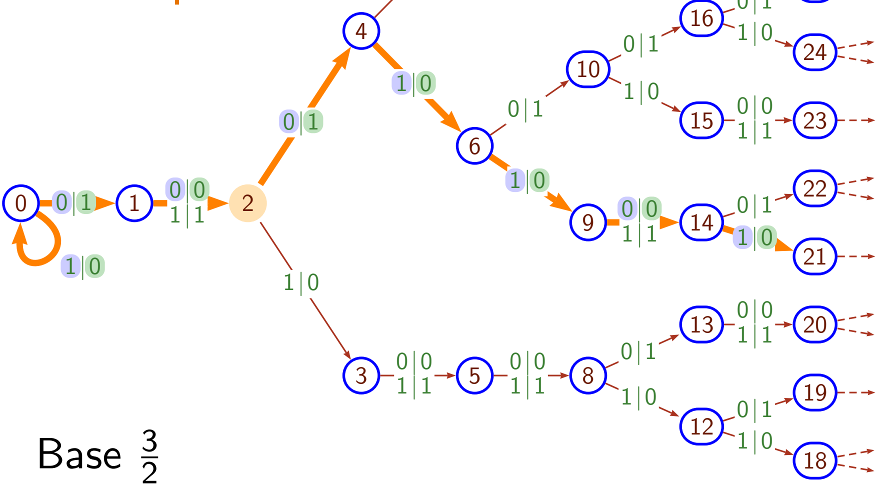

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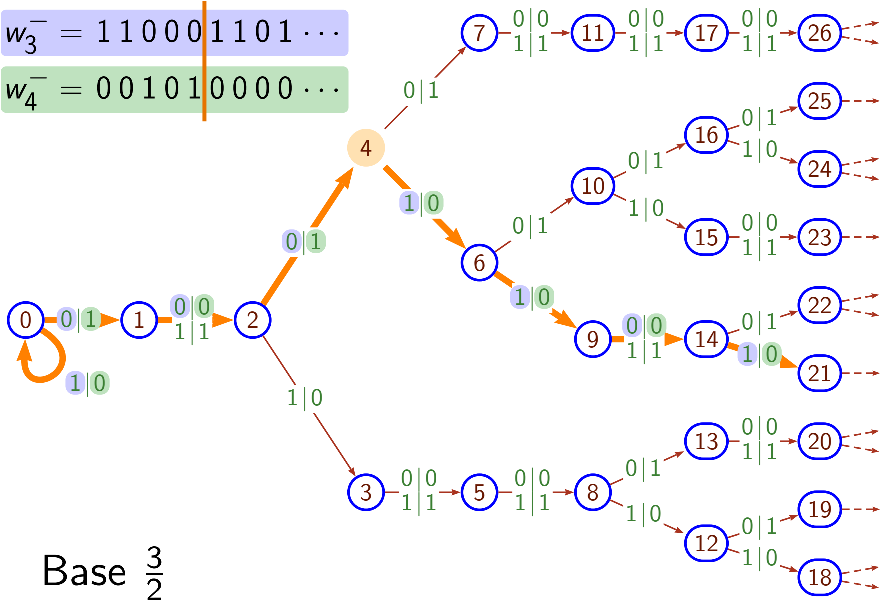

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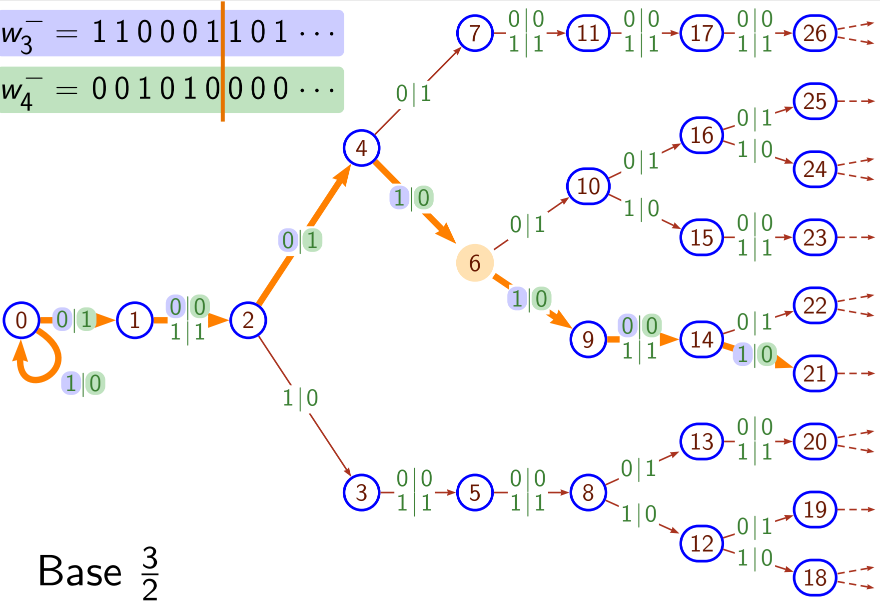

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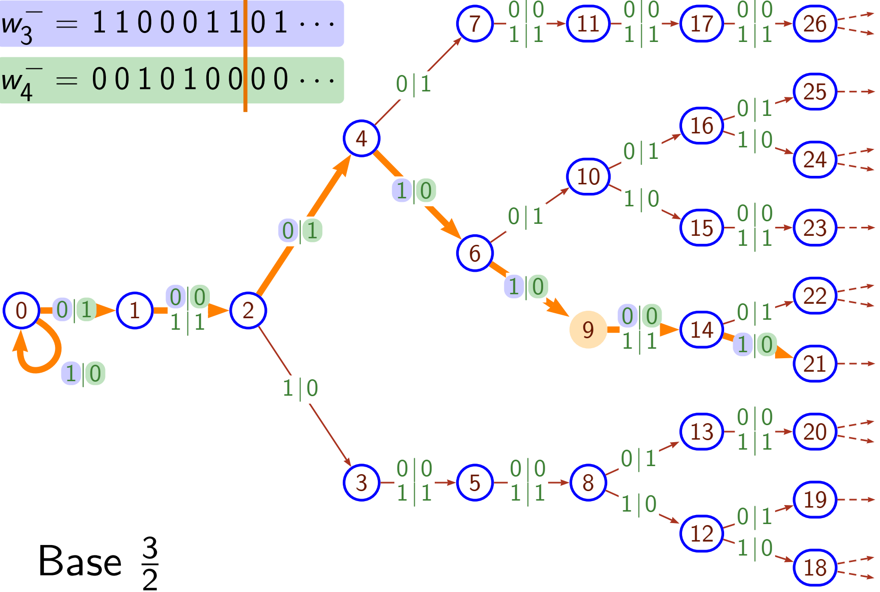

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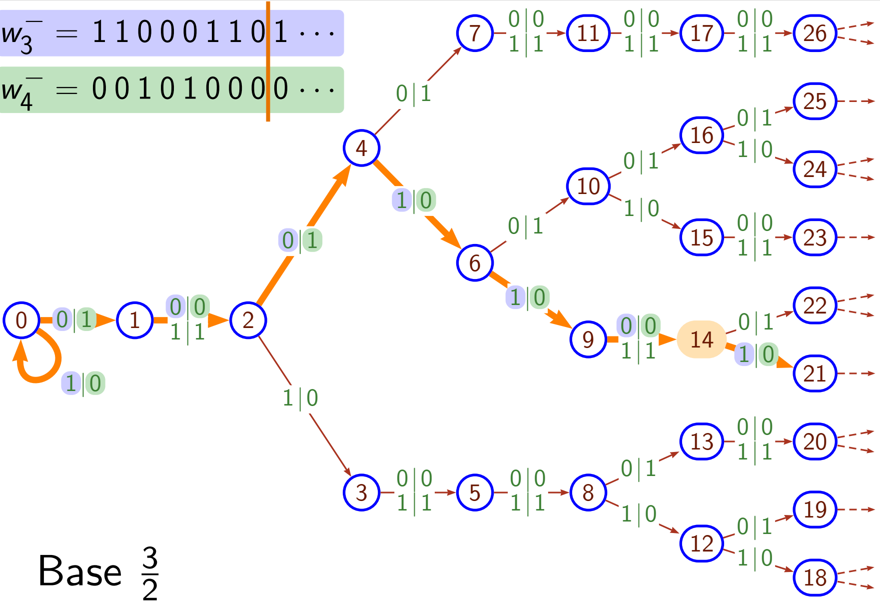

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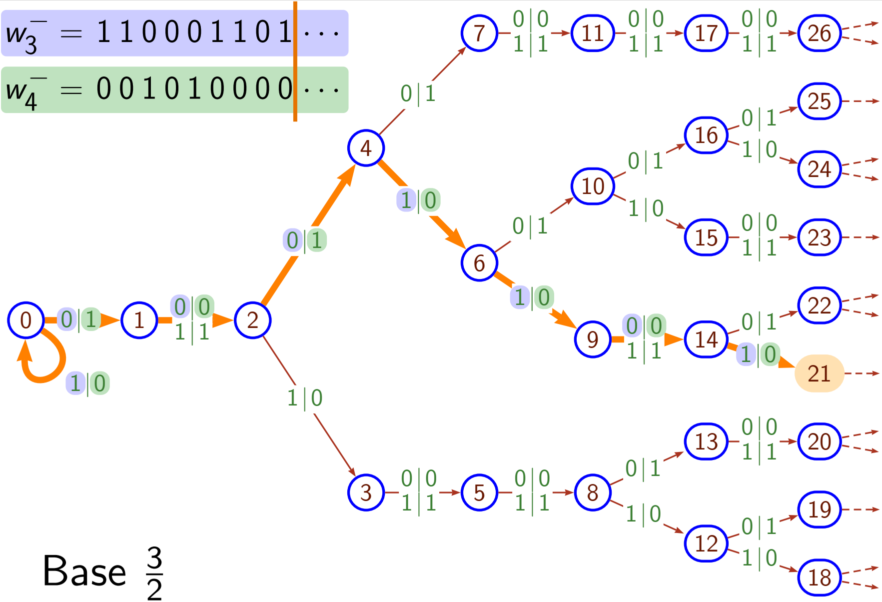

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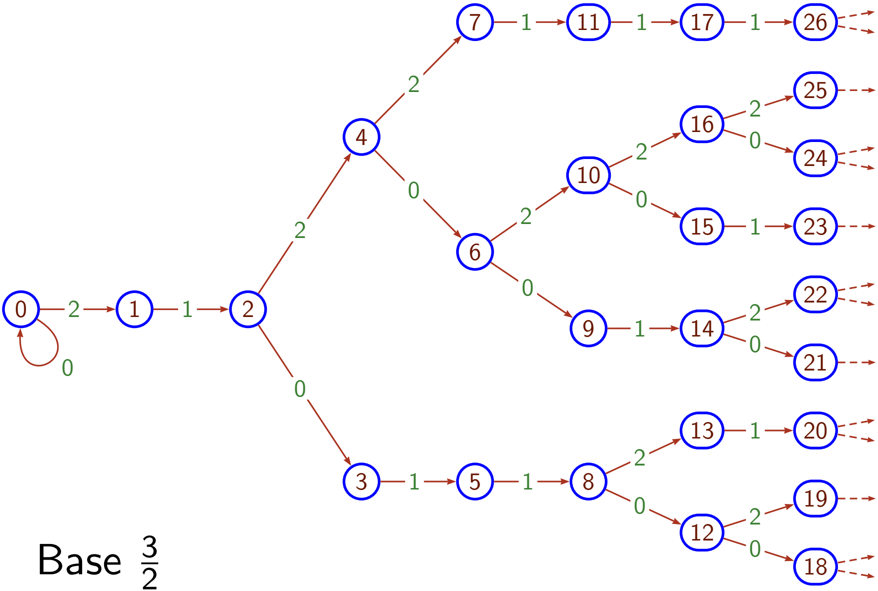


Base $\frac{3}{2}$

Figure: The infinite automaton $\mathcal{T}_{\frac{3}{2}}$

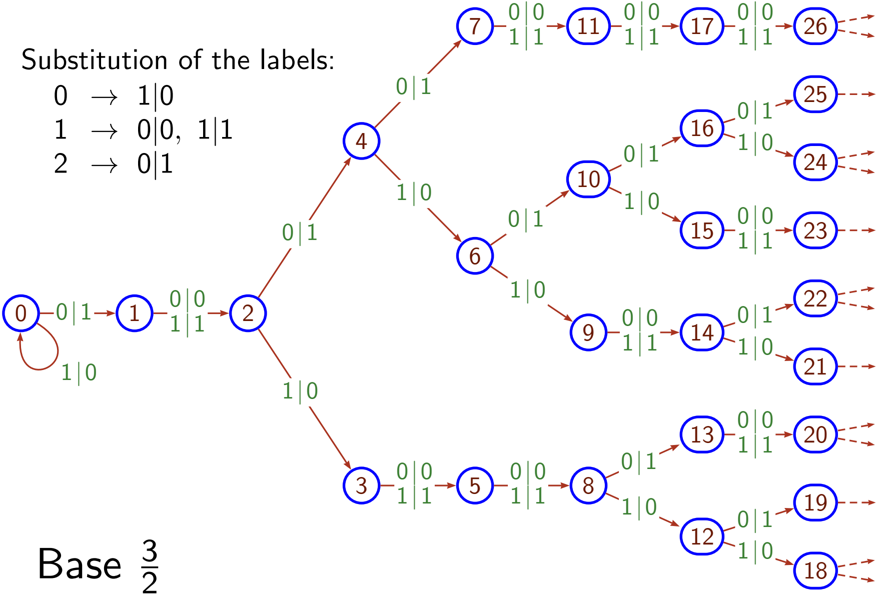
Example 1: the remarkable case of base $\frac{3}{2}$ (2)

Substitution of the labels:

$0 \rightarrow 1|0$

$1 \rightarrow 0|0,\ 1|1$

$2 \rightarrow 0|1$



Base $\frac{3}{2}$

Figure: $\mathcal{D}_{\frac{3}{2}}$, the infinite transducer realising ξ in base $\frac{3}{2}$

Proposition

If $\boxed{p = 2q - 1}$,

- the underlying graph of $D_{\frac{p}{q}}$ and $T_{\frac{p}{q}}$ are identical;
- the labels of the transitions of $D_{\frac{p}{q}}$ are obtained by an (injective) substitution from those of $T_{\frac{p}{q}}$.

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Theorem

The structure of $D_{\frac{p}{q}}$ is "very close" to the one of $\mathcal{T}_{\frac{p}{q}}$.

From $\mathcal{T}_{\frac{p}{q}}$ to $\mathcal{D}_{\frac{p}{q}}$ – Step 1: changing the alphabet

New alphabet:$B_{p,q} = \{p - (2q-1), \ldots, p-1\}$:

- $B_{p,q}$ always has $(2q-1)$ consecutive elements
- The maximal element of A_p and $B_{p,q}$ are the same.

From $\mathcal{T}_{\frac{p}{q}}$ to $\mathcal{D}_{\frac{p}{q}}$ – Step 1: changing the alphabet

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- if $p = (2q-1)$, $A_p = B_{p,q}$
- if $p < (2q-1)$, (we say that the base $\frac{p}{q}$ is "small")
  - $A_p \subseteq B_{p,q}$
  - Negative digits are added to A_p
- if $p > (2q-1)$, (we say that the base $\frac{p}{q}$ is "big")
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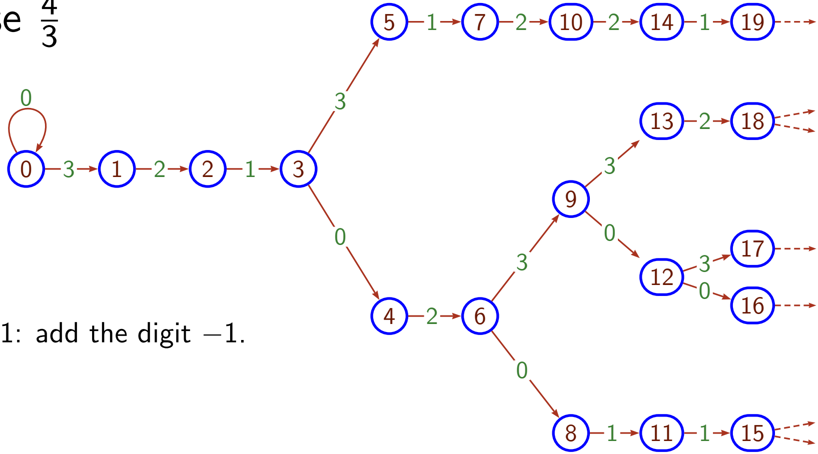
Obtained automaton: $\widehat{\mathcal{T}_{\frac{p}{q}}}$

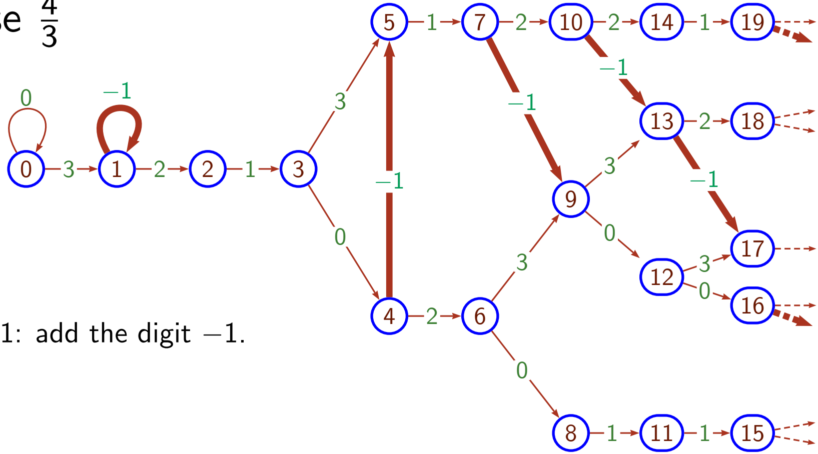
$$\forall a \in \mathbf{B_{p,q}} \quad \forall n, m \in \mathbb{N} \quad n \xrightarrow{a} m \iff n\,p + a = q\,m$$

Base $\frac{4}{3}$



Step 1: add the digit -1.

Base $\frac{4}{3}$



Step 1: add the digit −1.

From $\mathcal{T}_{\frac{p}{q}}$ to $\mathcal{D}_{\frac{p}{q}}$ – Step 2: substituting the label

$$\sigma : \quad B_{p,q} \longrightarrow \mathbb{P}(A_q \times A_q)$$

$$b \longmapsto \left\{ (a \,|\, a') \in A_q \times A_q \;\middle|\; (a' - a) = \underbrace{b - (\overbrace{p - q}^{\text{'center' of } B_{p,q}})}_{\text{signed distance to the center of } B_{p,q}} \right\}$$

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Eg., in base $\frac{4}{3}$, the substitution is:

σ: -1 $\mapsto \{\, 2\,|\,0 \,\}$

0 $\mapsto \{\, 1\,|\,0,\ 2\,|\,1 \,\}$

1 $\mapsto \{\, 0\,|\,0,\ 1\,|\,1,\ 2\,|\,2 \,\}$

2 $\mapsto \{\, 0\,|\,1,\ 1\,|\,2 \,\}$

3 $\mapsto \{\, 0\,|\,2 \,\}$

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1 $\mapsto \{\ 0\,|\,0,\ 1\,|\,1,\ 2\,|\,2\ \}$

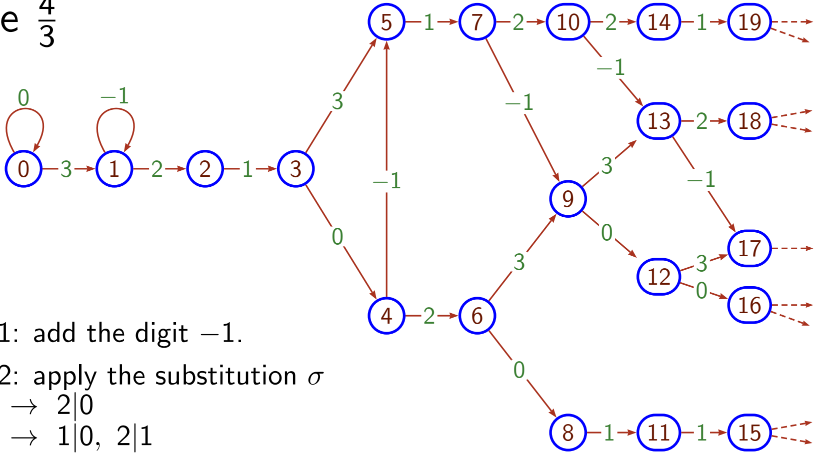
2 $\mapsto \{\ 0\,|\,1,\ 1\,|\,2\ \}$

3 $\mapsto \{\ 0\,|\,2\ \}$

- $B_{4,3} = \{-1, 0, 1, 2, 3\}$
- The center is 1
- $\Rightarrow$ '0' is equal to the center -1
- $\Rightarrow \sigma(0) = \{$ pairs of the form $(a\,|\,a{-}\mathbf{1})\ \}$

Example of the ‘small’ base $\frac{4}{3}$ – Step 2

Base $\frac{4}{3}$



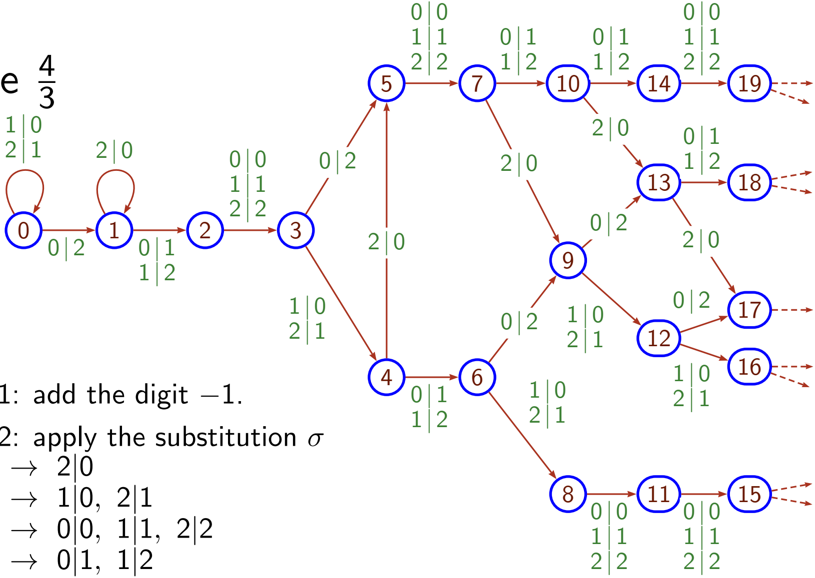
Step 1: add the digit -1.

Step 2: apply the substitution σ

$$
\begin{array}{rcl}
-1 & \to & 2|0 \\
0 & \to & 1|0,\ 2|1 \\
1 & \to & 0|0,\ 1|1,\ 2|2 \\
2 & \to & 0|1,\ 1|2 \\
3 & \to & 0|2
\end{array}
$$

Example of the ‘small’ base $\frac{4}{3}$ – Step 2

Base $\frac{4}{3}$



Step 1: add the digit -1.

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$-1 \rightarrow 2|0$

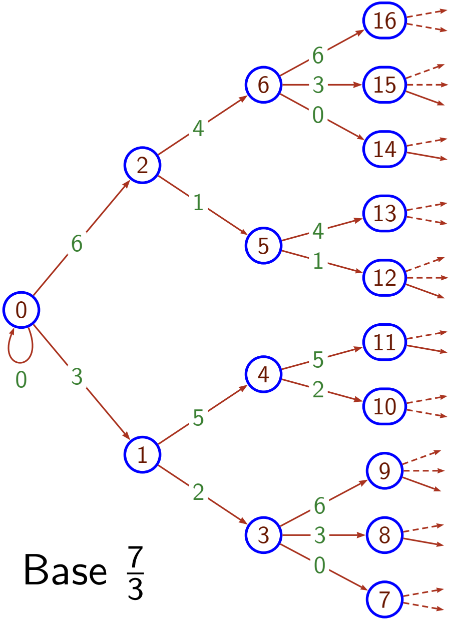
$0 \rightarrow 1|0,\ 2|1$

$1 \rightarrow 0|0,\ 1|1,\ 2|2$

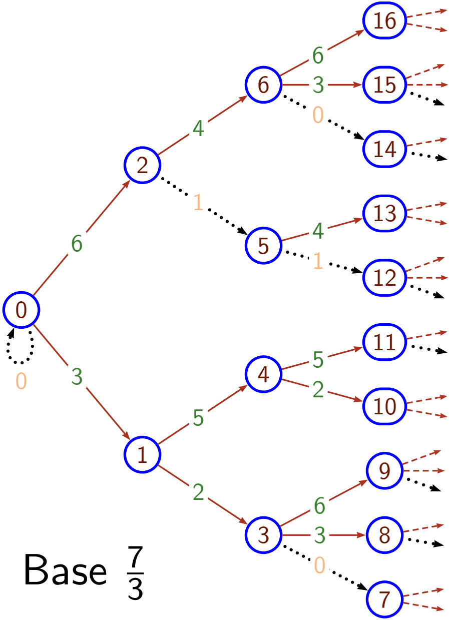
$2 \rightarrow 0|1,\ 1|2$

$3 \rightarrow 0|2$

Example of the ‘big’ base $\frac{7}{3}$

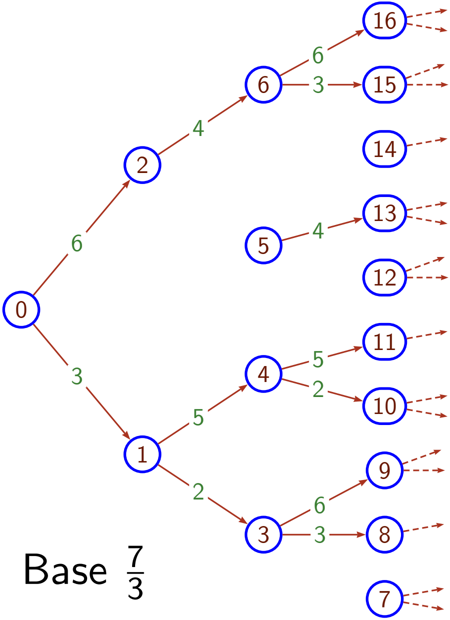


Base $\frac{7}{3}$



Step 1: delete $0's$ and $1's$.

Base $\frac{7}{3}$



Base $\frac{7}{3}$

Step 1: delete $0's$ and $1's$.

Step 2: apply the substitution σ:

$$
\begin{aligned}
2 &\rightarrow 2|0 \\
3 &\rightarrow 1|0,\ 2|1 \\
4 &\rightarrow 0|0,\ 1|1,\ 2|2 \\
5 &\rightarrow 0|1,\ 1|2 \\
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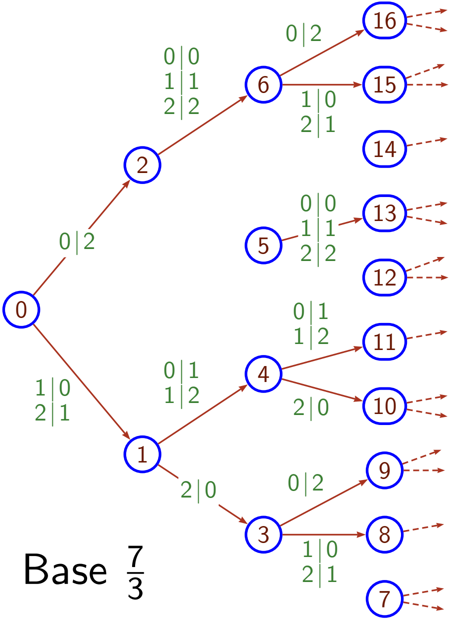
Example of the 'big' base $\frac{7}{3}$



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$$

Brief intuition of how it works

Overview

The label of the current state of $\mathcal{D}_{\frac{p}{q}}$ is the current difference between input and output.

Lemma

n, m, i, k: four node/integers
a, b: letters of A_q

$$\left.\begin{array}{c} w_n^- = a\, w_m^- \\ i \xrightarrow{a\,|\,b} k \ \text{ in } \mathcal{D}_{\frac{p}{q}} \end{array}\right\} \implies w_{n+i+1}^- = b\, w_{m+k+1}^-$$

	remaining input	remaining output
At state i:	w_n^-	w_{n+i+1}^-
At state k:	w_m^-	w_{m+k+1}^-

Plan

Fractal view of $W_{\frac{p}{q}}$

Reminder

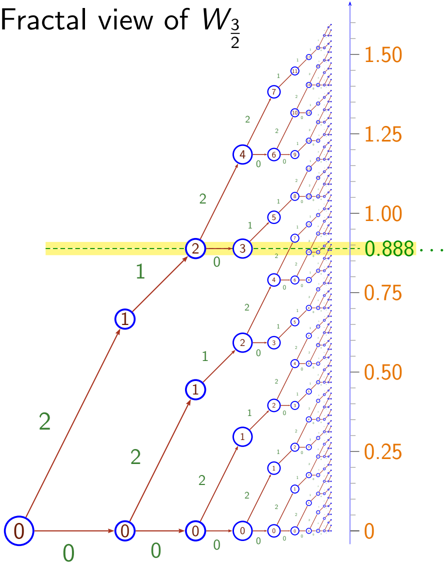
$$\rho_{\frac{p}{q}}(a_1 a_2 \cdots a_k \cdots) = \sum_{k\geq 1} \frac{a_k}{q}\left(\frac{p}{q}\right)^{-k}$$

$$\rho_{\frac{p}{q}}(a_1 a_2 \cdots a_k) = \rho_{\frac{p}{q}}(a_1 a_2 \cdots a_k 0^\omega)$$

Eg: $\rho_{\frac{3}{2}}(21) = 2\frac{1}{2}\frac{2}{3} + 1\frac{1}{2}\frac{4}{9} = 0.888\cdots$

$$\rho_{\frac{3}{2}}(210) = \rho_{\frac{p}{q}}(21) = 0.888\cdots$$

Fractal view of $W_{\frac{3}{2}}$

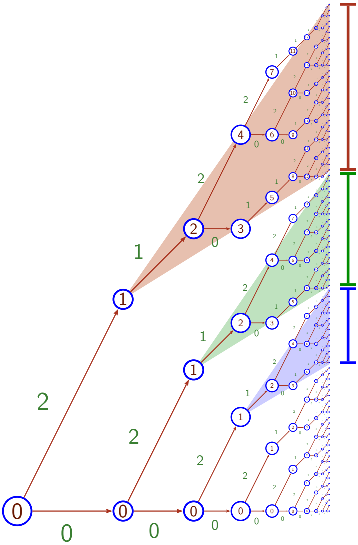


Span

Definition – span of the node X

The length of the interval reachable from X in the tree.

Example of Spans



$\text{span}(\text{"2"})\ (=\ \text{span}(\text{"21"}))$

$\text{span}(\text{"02"})\ =\ \text{span}(\text{"2"}) \times \frac{2}{3}$

$\text{span}(\text{"002"})\ =\ \text{span}(\text{"2"}) \times \left(\frac{2}{3}\right)^2$

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The length of the interval reachable from X in the tree.

Problems

- Spans decrease exponentially with depth.
- Only finitely many spans are above any positive bound.

Renormalised Span

Definition – renormalised span of the node X

the span of X multiplied by $\left(\frac{p}{q}\right)^k$, where k is the depth of X.

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$\Rightarrow$ Two nodes with the same label have the same renormalised span.

Notation

$\texttt{rspan}(n)$: the renormalised span of any node labelled by n.
$S_{\frac{p}{q}}$ denotes the set of the renormalised span of every node.

Topological Property of $S_{\frac{p}{q}}$

Theorem

- If $p \leq 2q - 1$, $\rho_{\frac{p}{q}}(S_{\frac{p}{q}})$ is dense.
- If $p > 2q - 1$, $\rho_{\frac{p}{q}}(S_{\frac{p}{q}})$ is nowhere dense.

Span words

Definition

We call **span-word** of n the word $(w_n^+ \ominus w_n^-)$ where

- where w_n^+ is the *maximal* word starting from n
- and "$\ominus$" denotes the digit-wise subtraction.
  (Example : $321 \ominus 012 = 31(-1)$)

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Properties

- The renorm. span of n is the evaluation of its span-word:
$$\forall n \in \mathbb{N} \quad \mathtt{rspan}(n) = \rho_{\frac{p}{q}}(w_n^+ \ominus w_n^-) \quad .$$
- Span-words belong to ${B_{p,q}}^{\omega}$.

Proposition

$\widehat{\mathcal{T}}_{\frac{p}{q}}$ accepts the topological closure of the language of the span-words.

Proof consists in simple arithmetic calculation.

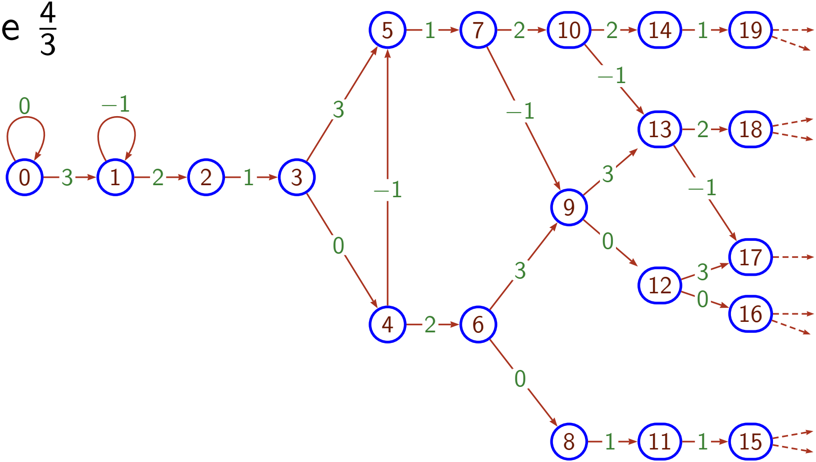
Reminder: $\widehat{\mathcal{T}}_{\frac{p}{q}}$ is a intermediary step of the construction $\mathcal{T}_{\frac{p}{q}} \to \mathcal{D}_{\frac{p}{q}}$.

- If $p \leq (2q-1)$ digits are added to $\mathcal{T}_{\frac{p}{q}}$, possibly none.
- If $p > (2q-1)$ at least one digit is removed from $\mathcal{T}_{\frac{p}{q}}$.

Sketch of proof of Theorem: case $p \leq (2q-1)$

Adding the digit -1 to $\mathcal{T}_{\frac{4}{3}}$

Base $\frac{4}{3}$



⇒ Using new digits does not produce new values.

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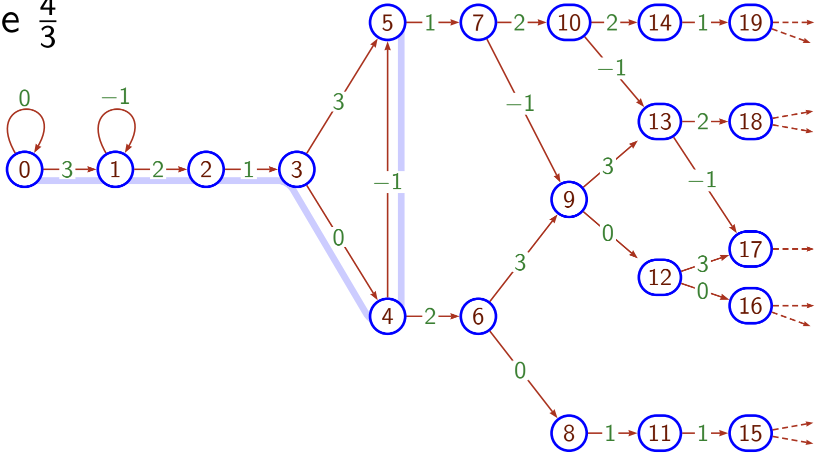

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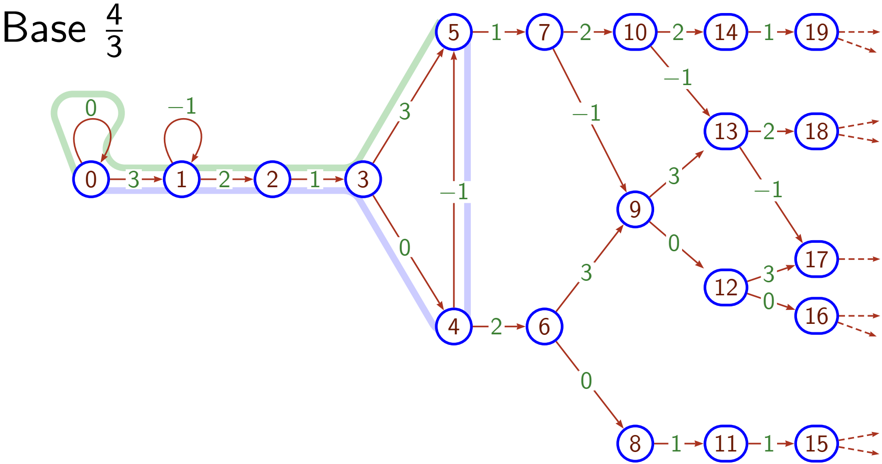

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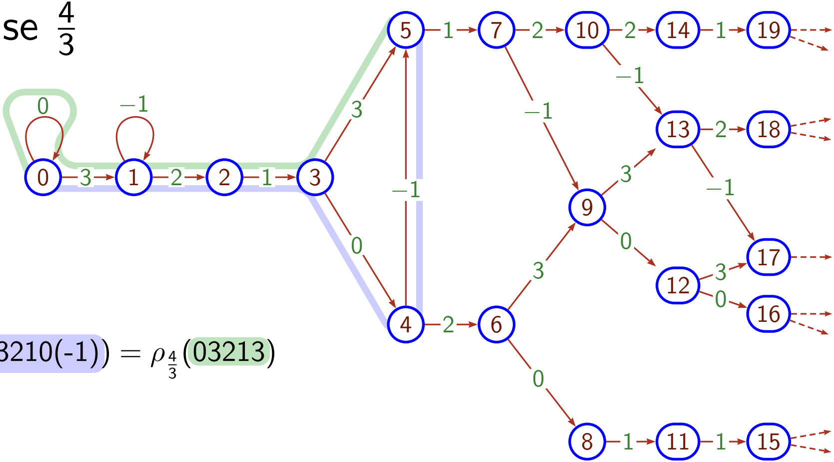

$\rho_{\frac{4}{3}}(3210(\text{-}1)) = \rho_{\frac{4}{3}}(03213)$

$\Rightarrow$ Using new digits does not produce new values.

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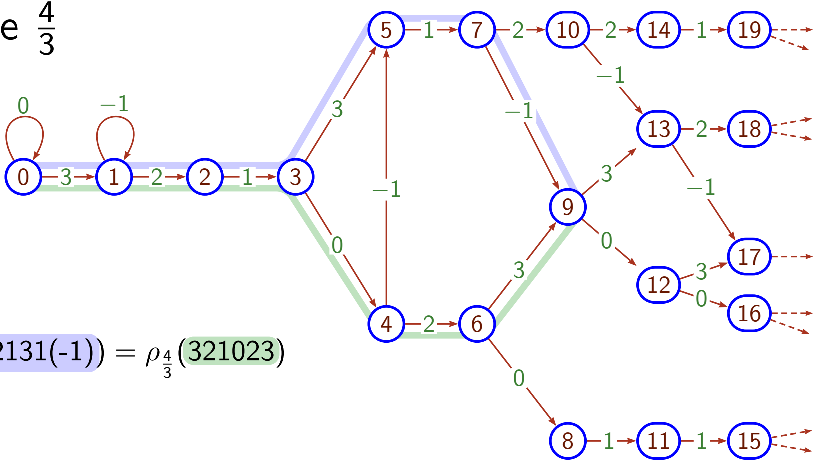
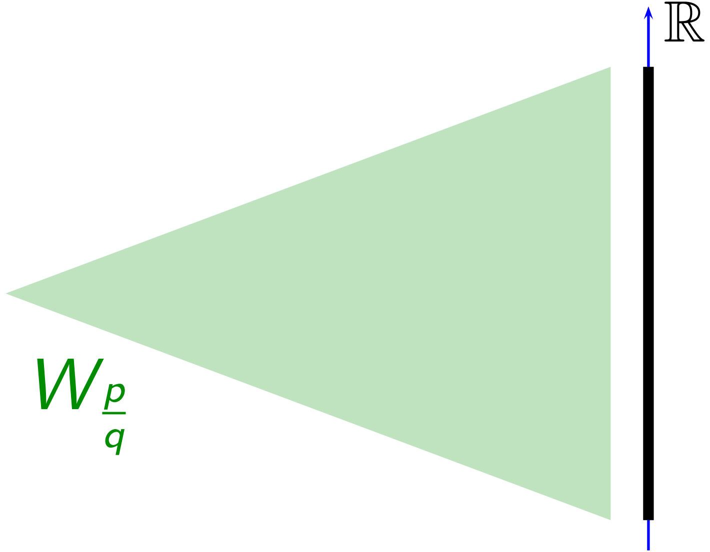


$\rho_{\frac{4}{3}}(32131(\text{-}1)) = \rho_{\frac{4}{3}}(321023)$

$\Rightarrow$ Using new digits does not produce new values.

Sketch of proof of Theorem: case $p > (2q-1)$

Deleting every letter a from $\mathcal{T}_{\frac{p}{q}}$ hence from the fractal tree $W_{\frac{p}{q}}$



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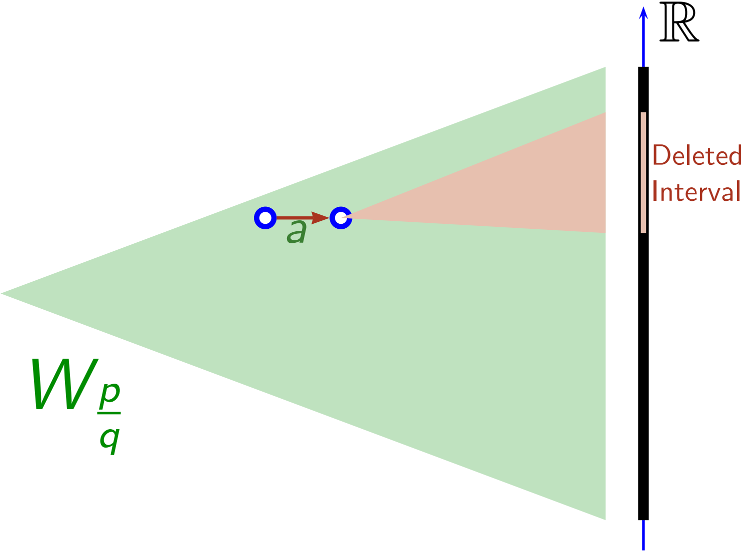

$W_{\frac{p}{q}}$

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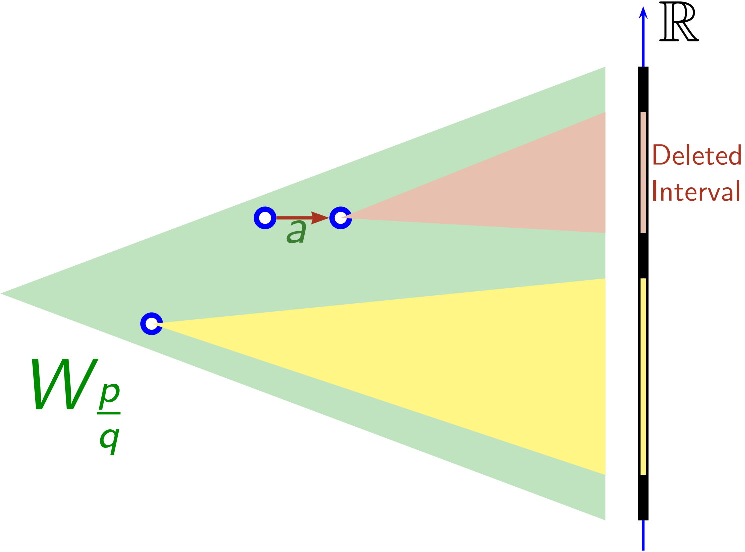

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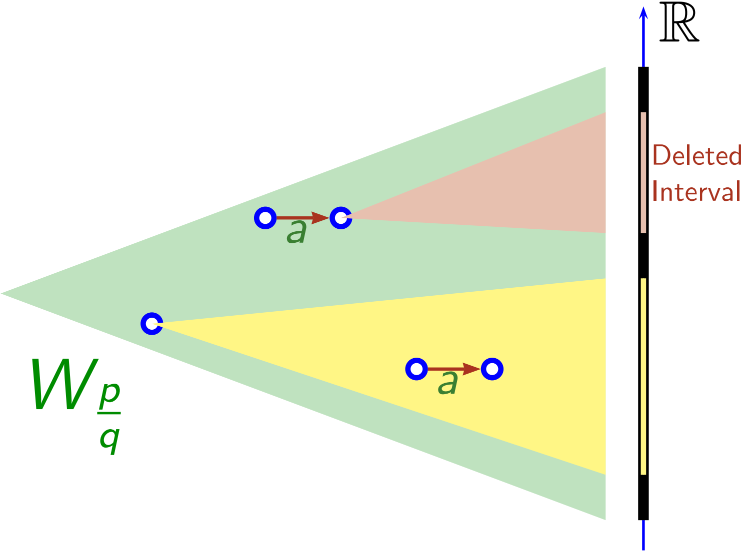

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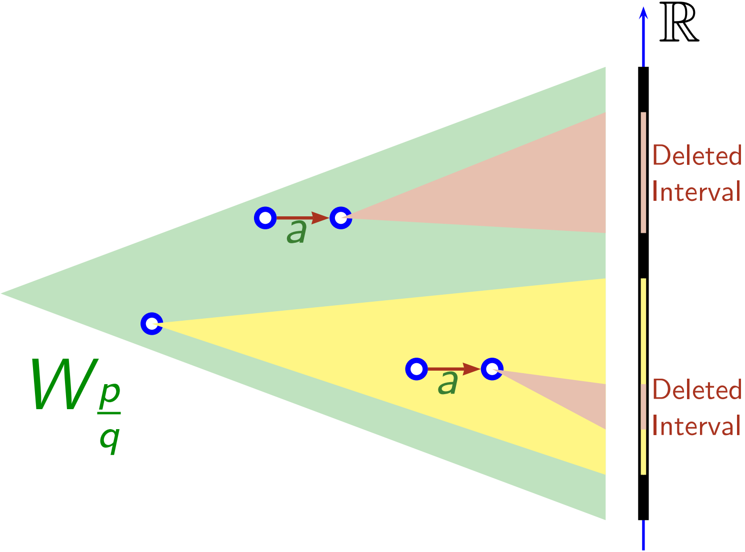

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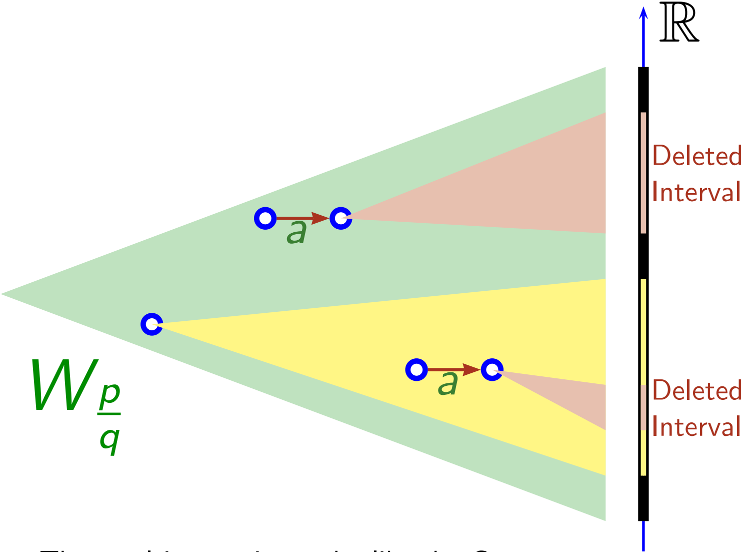

⇒ The resulting set is much alike the Cantor ternary set.

- $\mathcal{D}_{\frac{p}{q}}$ somehow requires the same structure as the original tree $\mathcal{T}_{\frac{p}{q}}$.
- The topological properties of the set of renorm. spans divides the rational base number systems in two classes.
- The cases $p = 2q - 1$ is remarkable in both constructions.

Next question

For a given integer n,
is there a *finite* transducer realising $w_n^- \mapsto w_{n+1}^-$?