Breadth-first signature of trees and rational languages

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joint work with Jacques Sakarovitch

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NS = Numeration system

Prefix-closed Abstract Rational NS (Lecomte–Rigo 2001)

Built from an arbitrary prefix-closed rational language.

Dumont-Thomas NS (Dumont-Thomas, 1989)

Built from an arbitrary substitution.

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Theorem

Every DTNS is a prefix-closed ARNS.

Every prefix-closed ARNS is easily† convertible to a DTNS.

† Through a finite, letter-to-letter and pure sequential transducer.

Outline

We call *tree* a...

Directed graph which is

- **Rooted:** a node is called *the root* (leftmost in the figures)
- **Directed outward from the root:** there is a unique path from the root to every other node.
- **Ordered:** the children of every node are ordered

  (In the figures, lower children are smaller.)

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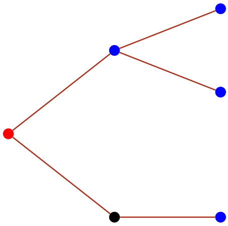
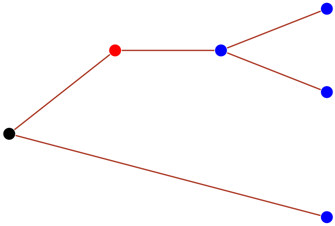


$\neq$



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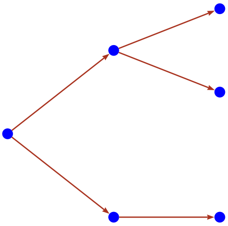

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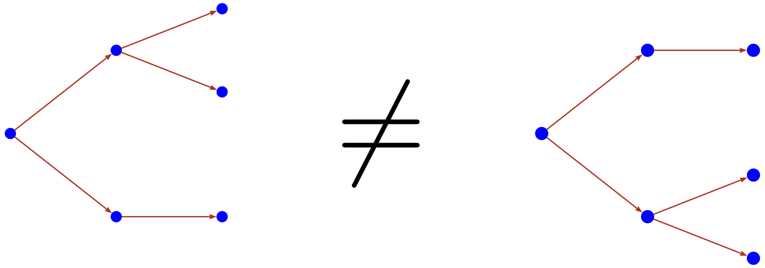
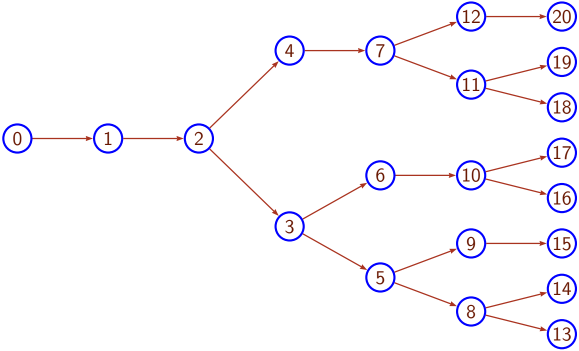


Every tree has a canonical breadth-first traversal



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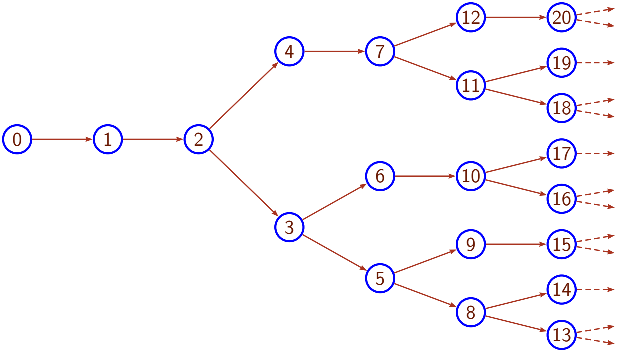

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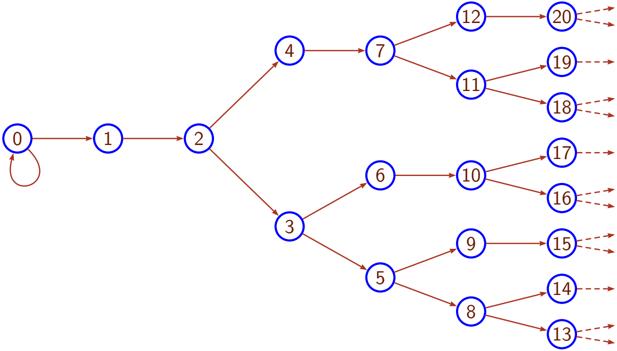
- For convenience, there is loop on the root.



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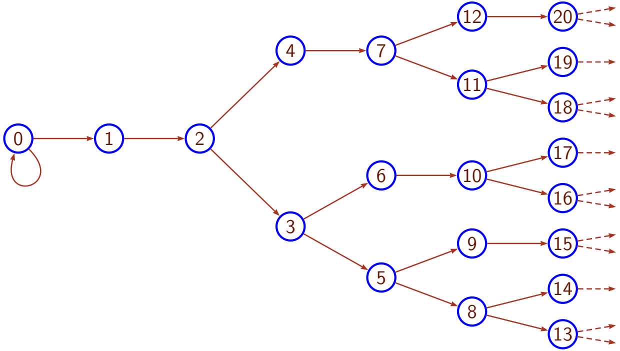
The **signature** of a tree is the sequence of the degree of the nodes taken in breadth-first order.



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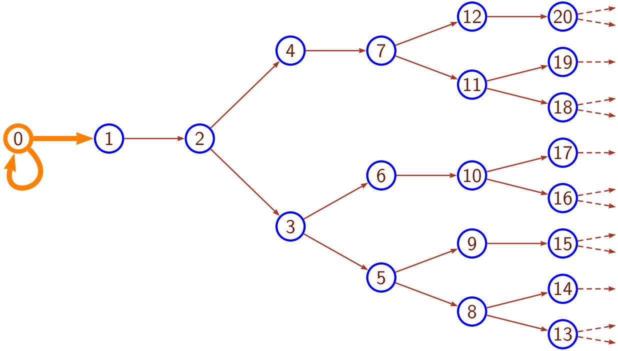

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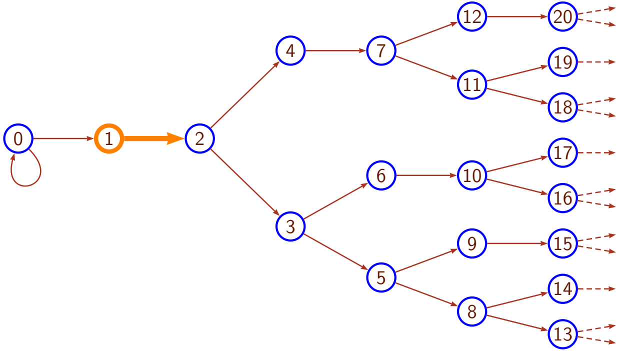

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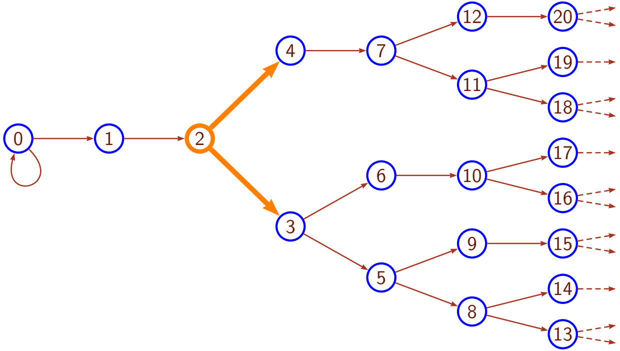

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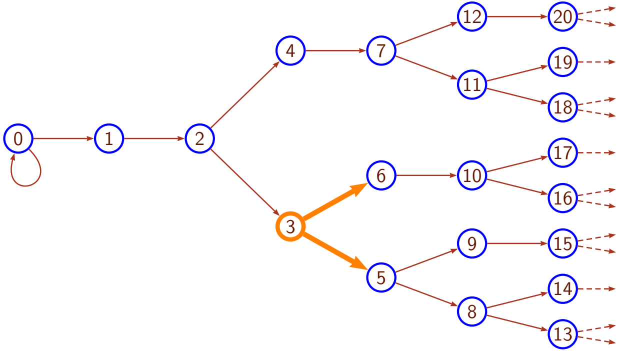

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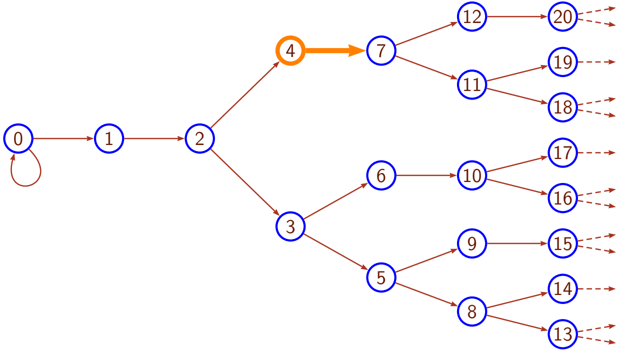

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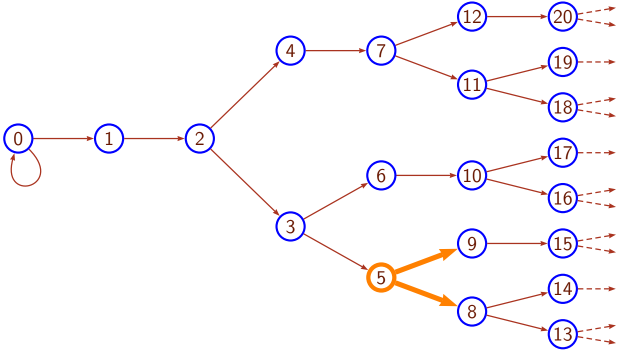

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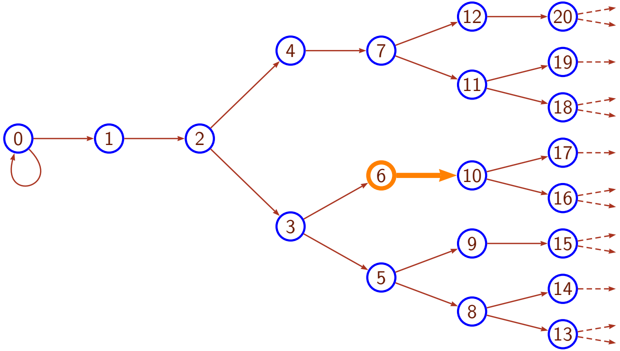

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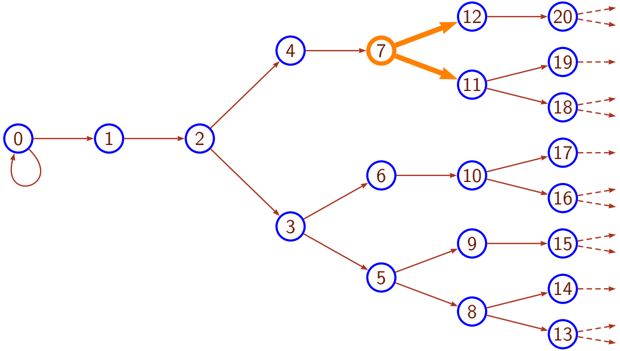

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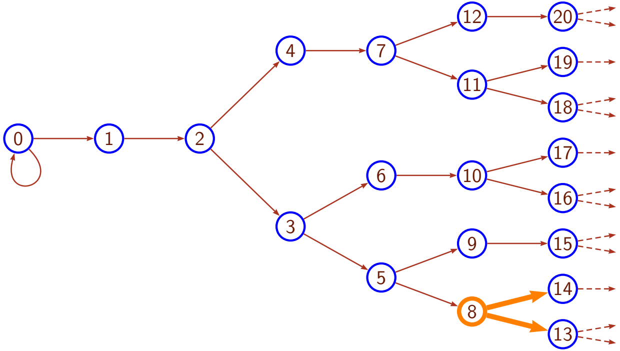

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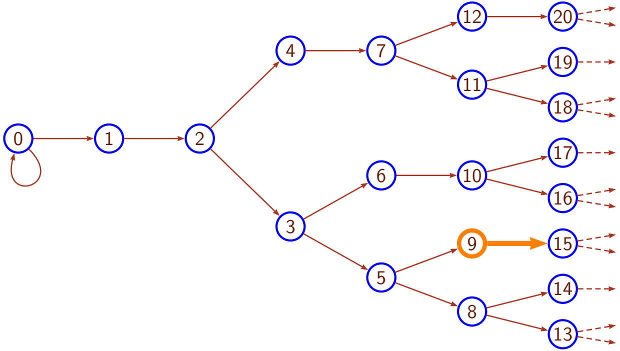

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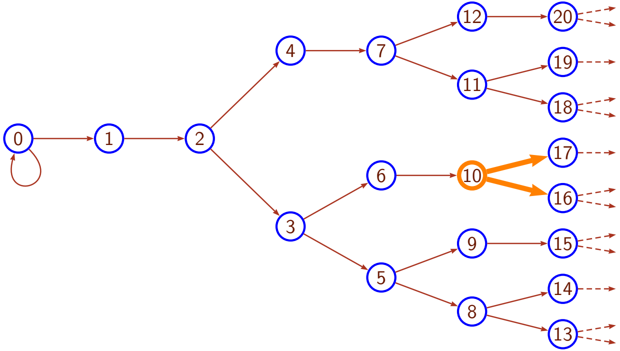

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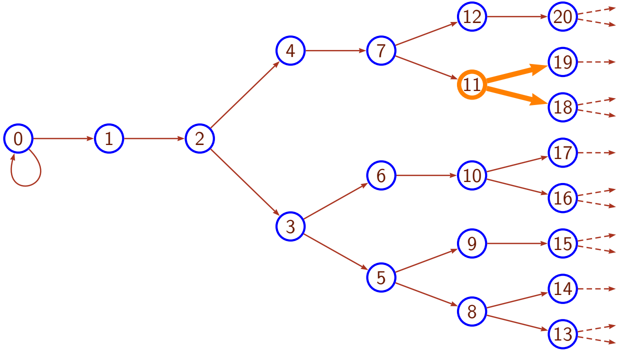

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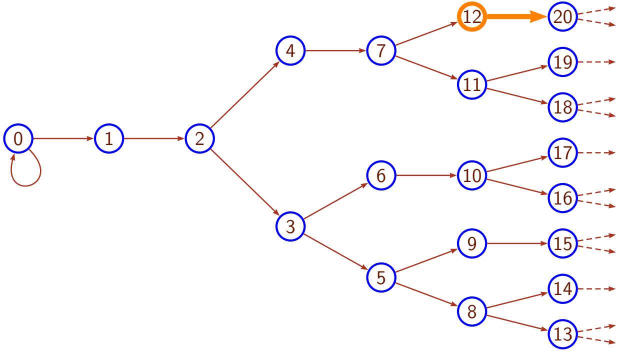

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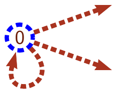
Signature is characteristic of tree

$$\mathbf{s} = (\; 3 \;\; 2 \;\; 1 \;)^{\omega}$$



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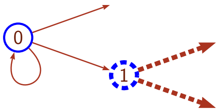

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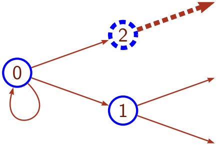

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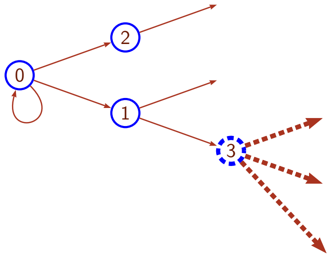

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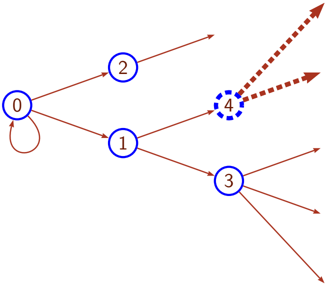

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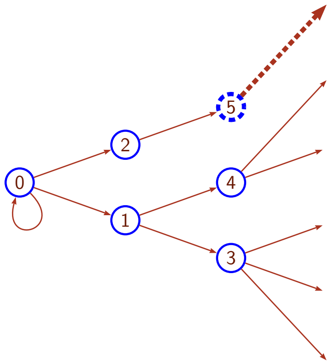

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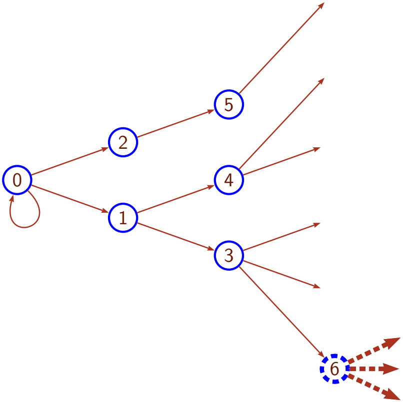

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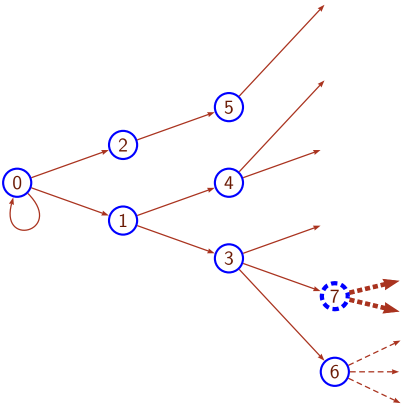

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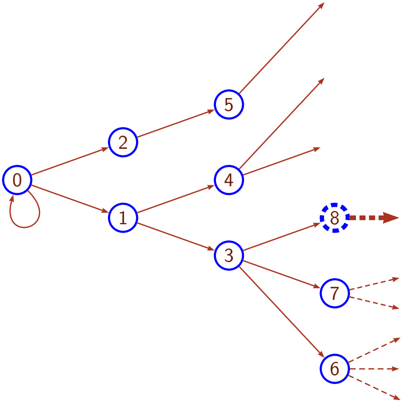

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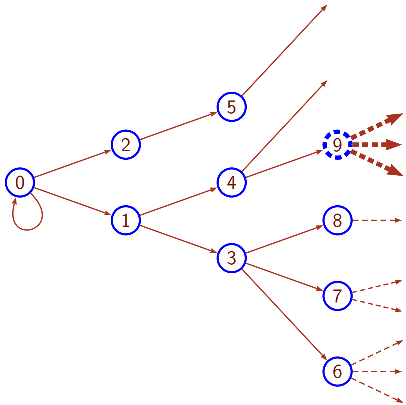

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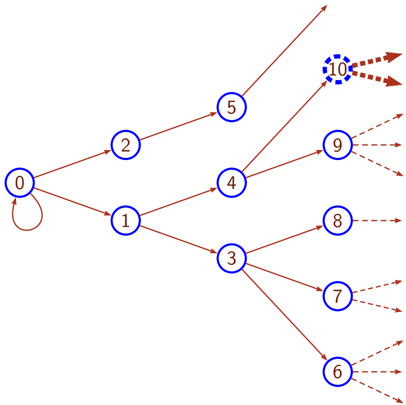

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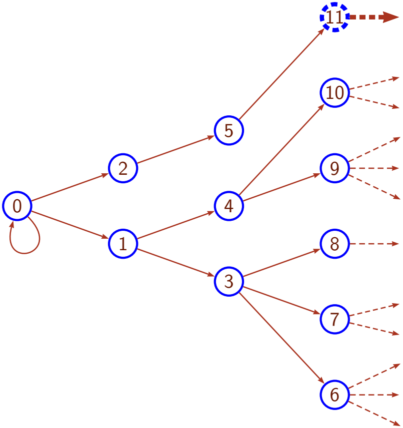
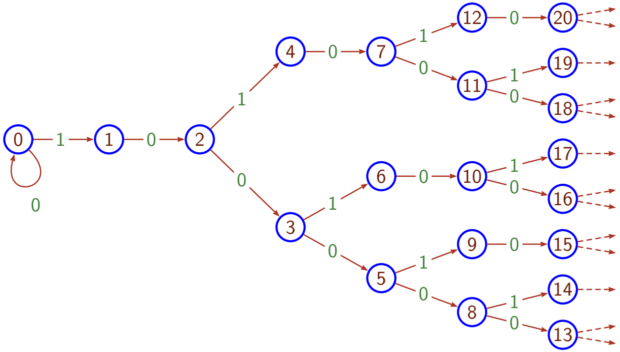


Prefix-closed languages and labelled trees

Alphabets are ordered hence

prefix-closed languages = labelled trees.



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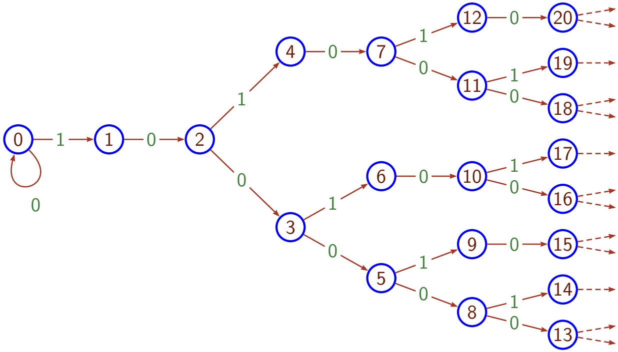

Figure : Integer representations in the Fibonacci numeration system.

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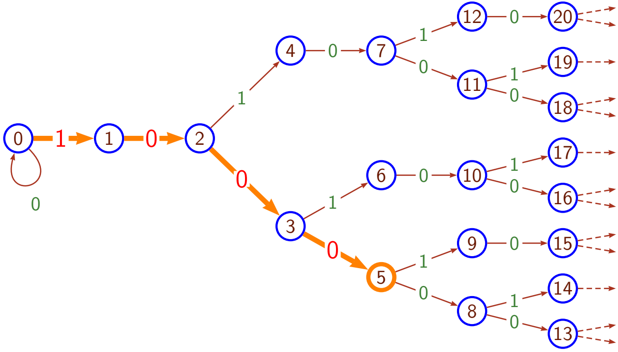

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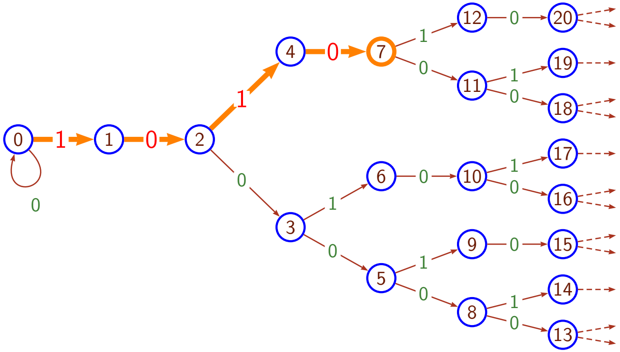

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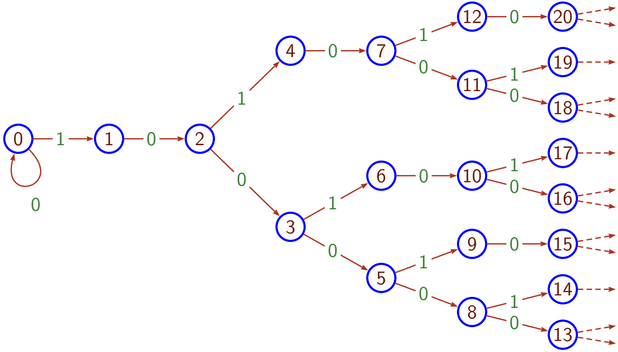

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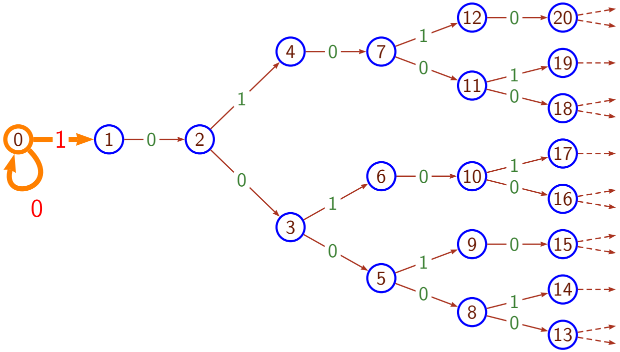

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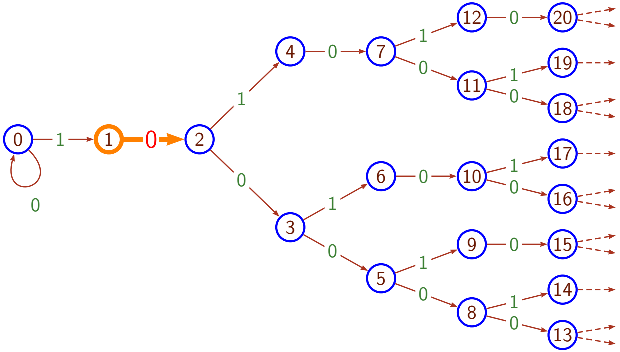

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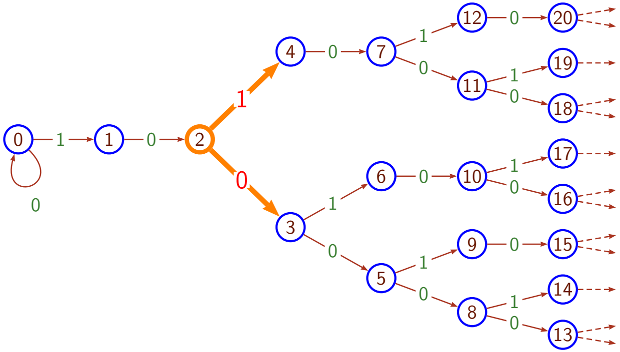

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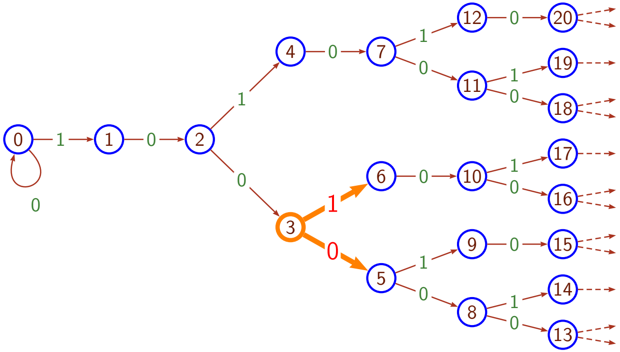

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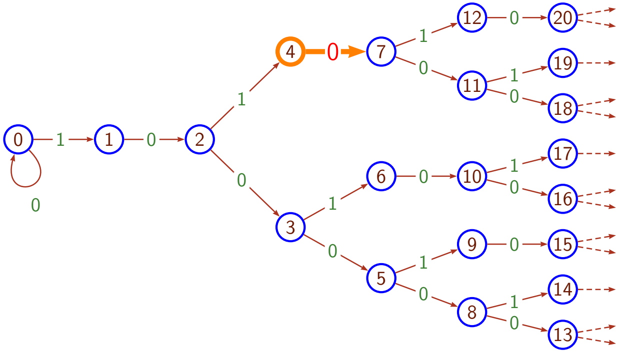

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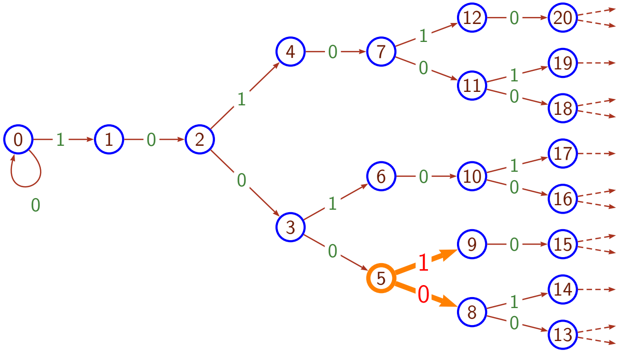

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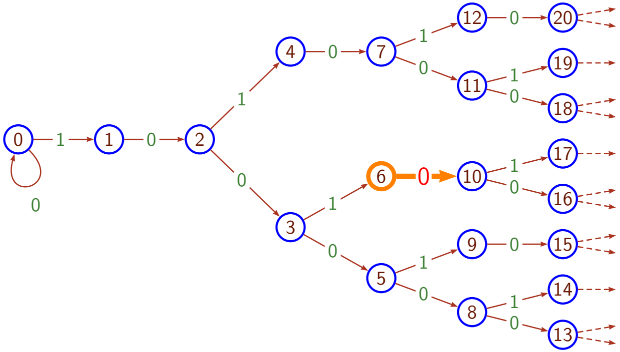

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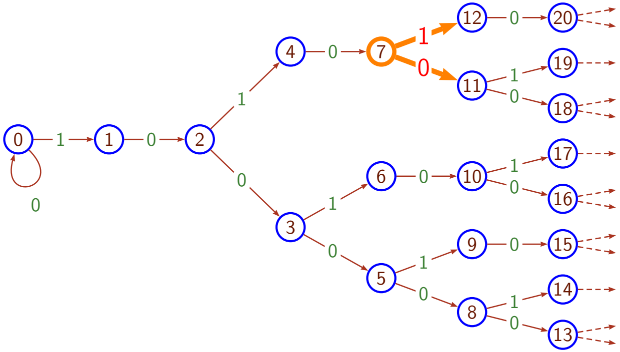

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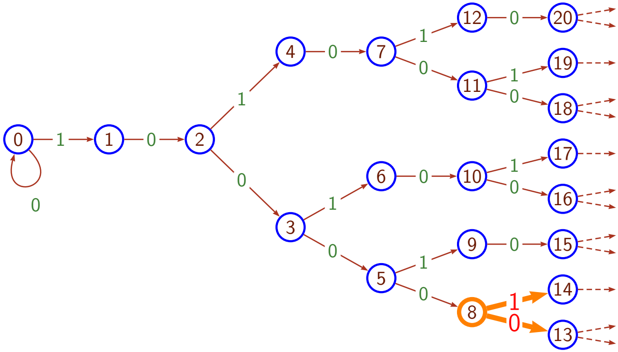

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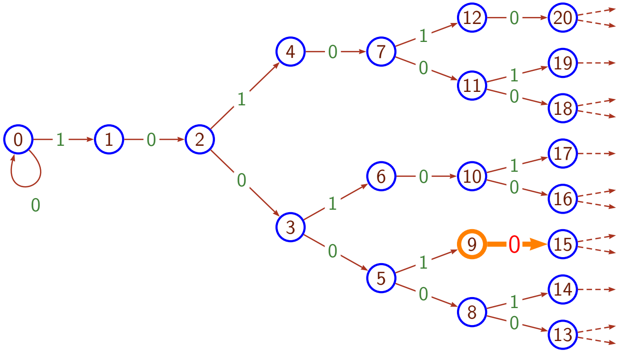

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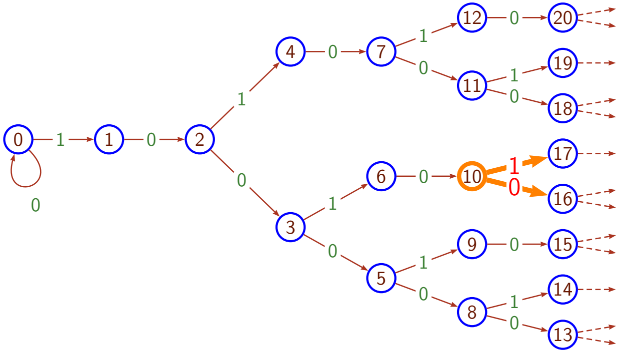

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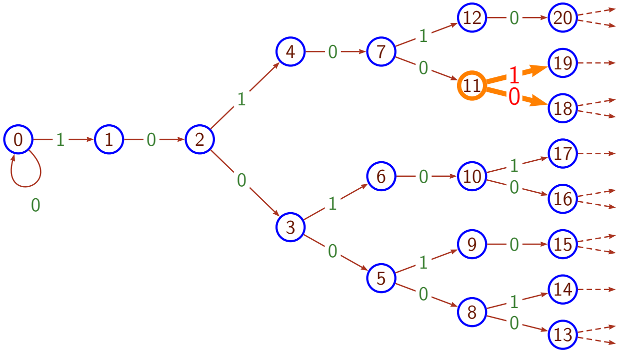

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Serialisation of a prefix-closed language

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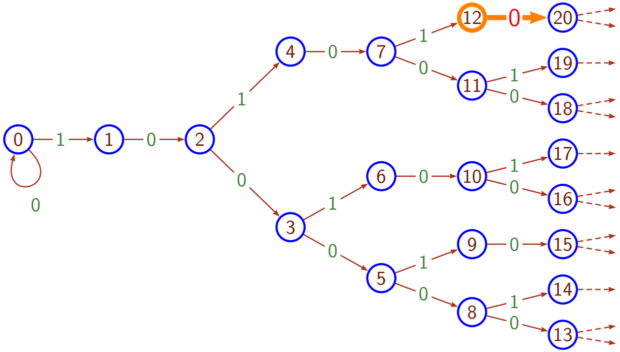

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The pair signature/labelling is characteristic

$$\mathbf{s} = (3\ 2\ 1)^\omega$$
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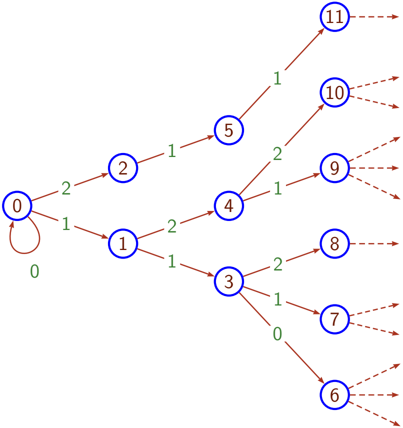

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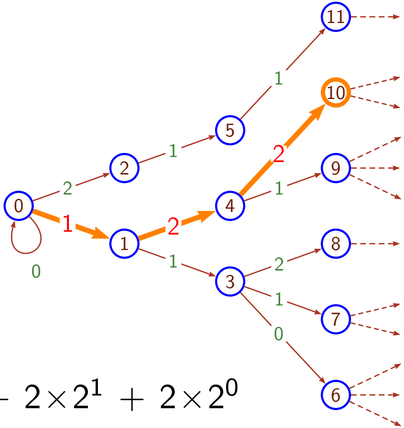

$$10 = 1\times 2^2 + 2\times 2^1 + 2\times 2^0$$

Figure : Non-canonical integer representations in base 2.

Theorem

L: a prefix-closed language.
Signature(L) is substitutive $\Leftrightarrow$ L is accepted by a finite automaton.

A word on substitution

A substitution σ is a morphism $A^* \to A^*$.

Running examples

Fibonacci substitution: $\{a, b\} \to \{a, b\}^*$
$a \mapsto ab$
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In this case, $\sigma^\omega(a)$ exists and is called a purely substitutive word.

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- $D \subset N$
- $\forall b,\ f_\sigma(b) = |\sigma(b)|$

We call $f_\sigma(\sigma^\omega(a))$ a **subtitutive signature**.

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If g is a morphism such that

- $\forall b, \; |g(b)| = |\sigma(b)|$
- if $g(b) = c_0 c_1 \cdots c_k$ then $c_0 < c_1 < \cdots < c_k$

We call $g(\sigma^\omega(a))$ a **substitutive labelling**.

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$$\sigma(a) = \text{ab} \implies f_\sigma(a) = 2$$
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This pair signature/labelling defines the language of integer representations in the Fibonacci numeration system.

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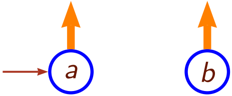

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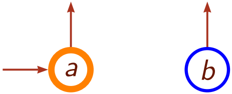

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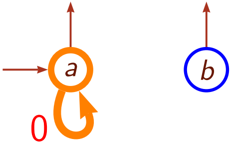

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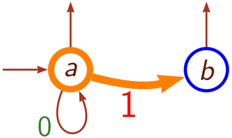

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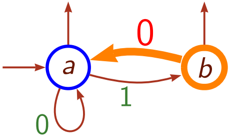

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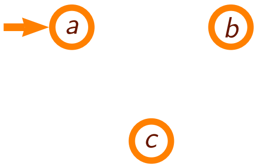

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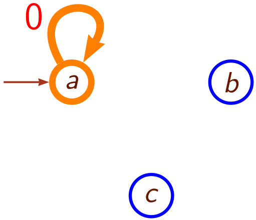

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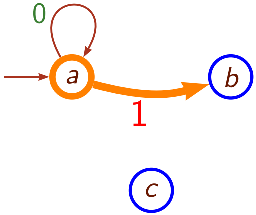

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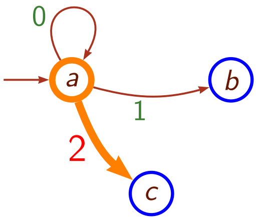

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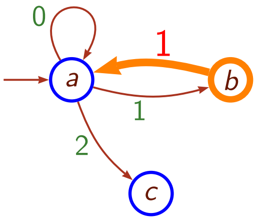

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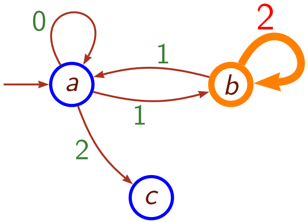

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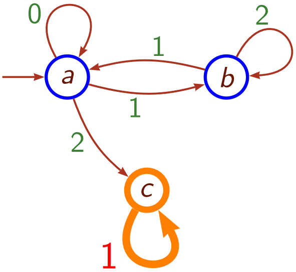

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Proof: Unfold the automaton $\mathcal{A}_{(\sigma,g)}$.

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The language accepted by $\mathcal{B}$ has signature $(\sigma_{\mathcal{B}}, g_{\mathcal{B}})$.

Follows directly from the other direction.

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Example: $2 <_{\text{rad}} 12 \qquad 12 <_{\text{rad}} 21$.

Abstract Numeration System (ANS, Lecomte-Rigo)

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Definition (ANS L)

L: language over an ordered alphabet A.
$\langle n \rangle_L$ is the $(n+1)$-th word of L in the radix order.

In our scheme, $\langle n \rangle_L$ is the word labelling the path $0 \longrightarrow n$.

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g_σ: morphism $A^* \to A_\sigma^*$

$g_\sigma(b) = [u_0]\,[u_1] \cdots [u_{k-1}]$

- $k = |\sigma(b)|$
- u_i is the prefix of length i of $\sigma(b)$

Example : $g_\sigma(a) = [\varepsilon]\,[a]\,[ab]$ $g_\sigma(b) = [\varepsilon]\,[a]$ $g_\sigma(c) = [\varepsilon]$

Dumont-Thomas automaton $\mathcal{A}_{(\sigma, g_\sigma)}$

$\sigma(a) = abc$
$\sigma(b) = ab$
$\sigma(c) = c$

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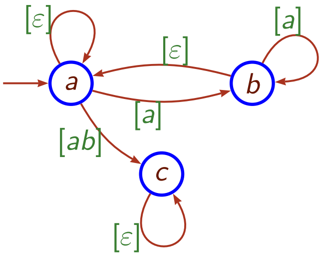

Dumont-Thomas Numeration System (DTNS)

Definition

ρ function ${A_\sigma}^* \to A^*$

$\rho([u_k] \ldots [u_2]\,[u_1]\,[u_0]) \;=\; \sigma^k(u_k)\sigma^{k-1}(u_{k-1})\cdots\sigma^2(u_2)\sigma(u_1)u_0$

Example: $\rho_\sigma([a]\,[\varepsilon]\,[ab]) \;=\; \sigma^2(a) \quad \sigma(\varepsilon) \quad ab$

$\qquad abc\,abc \quad \varepsilon \quad ab = \text{abc abc ab}$

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Theorem (Dumont Thomas '89)

$\forall n \in \mathbb{N}$

$\exists!$ word $[u_k]\ \dots\ [u_2]\,[u_1]\,[u_0]$ accepted by $\mathcal{A}_{(\sigma, g_\sigma)}$ such that

- $u_k \neq \varepsilon$
- $|\rho([u_k]\ \dots\ [u_2]\,[u_1]\,[u_0])| \ = \ n$

$[u_k]\ \dots\ [u_2]\,[u_1]\,[u_0]$ is the representation of n in the DTNS.

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Example: $[a]\,[\varepsilon]\,[ab]$ is the representation of 8 .

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Theorem

1. Every DTNS is a prefix-closed ARNS.

2. Every prefix-closed ARNS is easily† convertible to a DTNS.

† Through a finite, letter-to-letter and pure sequential transducer.

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Prefix-Closed ARNS L

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Prefix-Closed ARNS $L \longrightarrow$ Automaton $\mathcal{A} \longrightarrow$ Morphisms (σ, g)

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Prefix-Closed ARNS L of signature (s, λ_1) $\longrightarrow$ Automaton $\mathcal{A}$ $\longrightarrow$ Morphisms (σ, g)

where $s = f_\sigma(\sigma^\omega(a))$ $\lambda_1 = g(\sigma^\omega(a))$

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Prefix-closed ARNS K of signature (s, λ_2) $\longleftarrow$ DT Automaton $\mathcal{A}_{(\sigma, g_\sigma)}$ $\longleftarrow$ DTNS σ

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Proposition

L: prefix-closed ARNS of signature (s, λ_1)
K: prefix-closed ARNS of signature (s, λ_2)
The conversion function $\langle n \rangle_L \mapsto \langle n \rangle_K$ is very simple†.

$$\mathbf{s} \;=\; u\, r^{\omega} \qquad \text{with} \qquad r \;=\; r_0\, r_1\, r_2 \cdots r_{q-1}$$

Definition: growth ratio

$$\mathtt{gr}(\mathbf{s}) \;=\; \frac{r_0 + r_1 + \cdots + r_{q-1}}{q}$$

Other works: Ultimately periodic signatures

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Theorem (MS, to appear)

If $\mathtt{gr}(\mathbf{s}) \in \mathbb{N}$, then **s** generates the language of a finite automaton.
It is linked[‡] to the integer base $b = \mathtt{gr}(\mathbf{s})$.

If $\mathtt{gr}(\mathbf{s}) \notin \mathbb{N}$, then **s** generates a non-context-free language.
It is linked[‡] to the *rational base* $\frac{p}{q} = \mathtt{gr}(\mathbf{s})$. (cf. Akiyama et al. '08)

‡ It is a non-canonical representation of the integers (using extra digits).