Rational base number systems
BLIP languages and finitely generated monoids

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Outline

Integer Base

- base $p > 1$
- alphabet $A_p = \{0, 1, \cdots, p-1\}$

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- value $\pi(a_n \cdots a_1 a_0) = \sum_{i=0}^{n} a_i p^i$

Example (base 3) -

$$\begin{aligned} \pi(12) &= (3 \times 1) + (1 \times 2) &= 5 \\ \pi(122) &= (9 \times 1) + (3 \times 2) + (1 \times 2) &= 17 \end{aligned}$$

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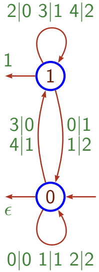
- value $\pi(a_n \cdots a_1 a_0) = \sum_{i=0}^{n} a_i p^i$
- $\pi(A_p^*) = \mathbb{N}$

- representation $\langle n \rangle_p = \langle n' \rangle_p.a$
  - (n', a) is the Euclidean division de n par p.
- $\langle \mathbb{N} \rangle_p = (A_p \backslash \{0\}) \cdot A_p^*$

Digit-wise addition : $A_p \times A_p \mapsto A_{2p-1}$
example (base 3) : 122+12 = 134

Alphabet conversion : $A_{2p-1} \mapsto A_p$



$$s \xrightarrow{a|b} t \iff s + a = pt + b$$

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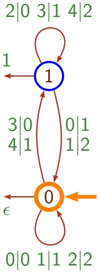

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$0 \longleftarrow$

Integer base - additionner

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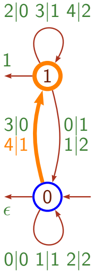

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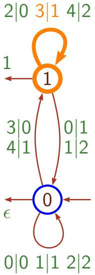

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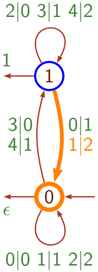

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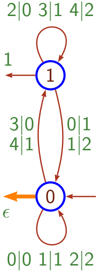

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Rational Base

- base $\frac{p}{q} > 1$ irreducible fraction ($p > q$ and $p \wedge q = 1$).

- representation $\langle n \rangle_{\frac{p}{q}} = \langle n' \rangle_{\frac{p}{q}}.a$:
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$2 \times 3 = 3 \times N_1 + a_0; \qquad \Rightarrow N_1 = 2$ and $a_0 = 0$.

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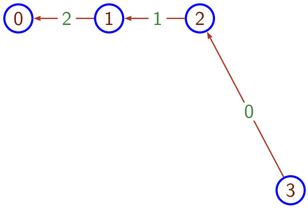
Computation tree of the representations in base $\frac{3}{2}$



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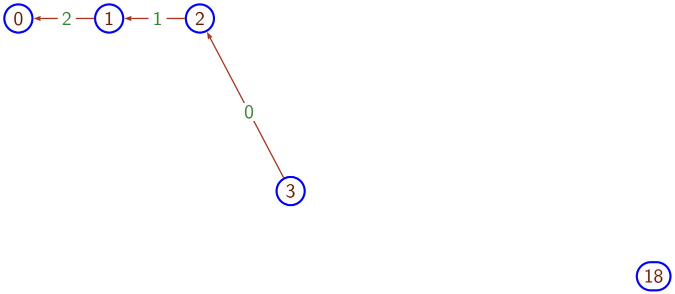

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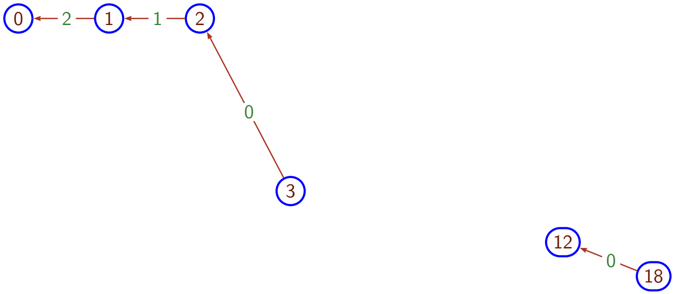

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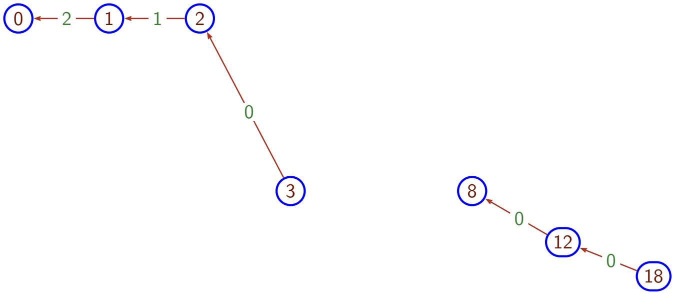

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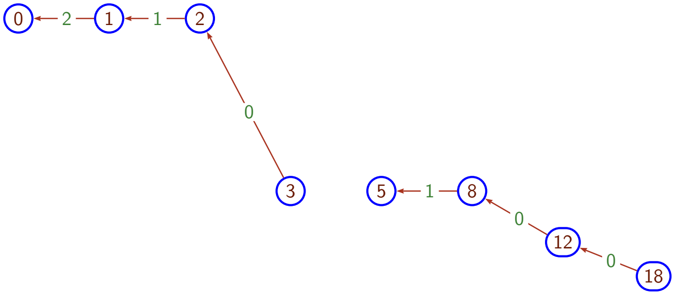

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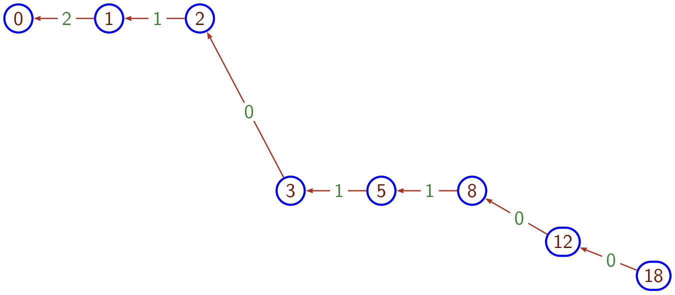

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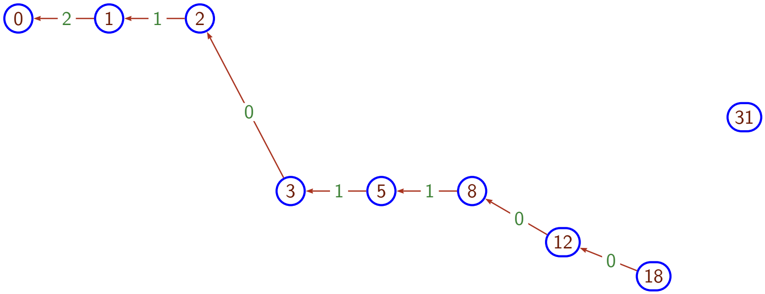

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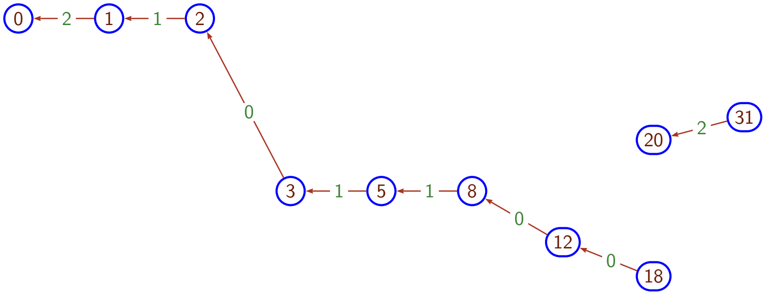

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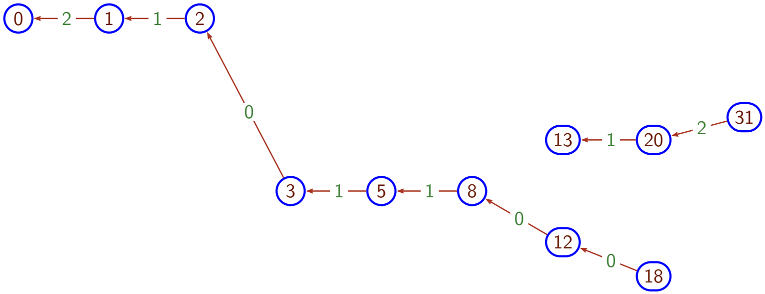

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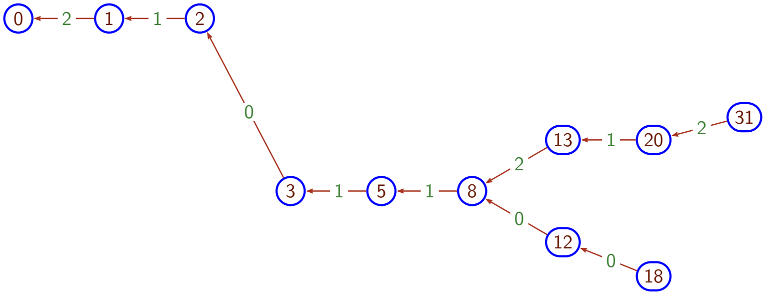

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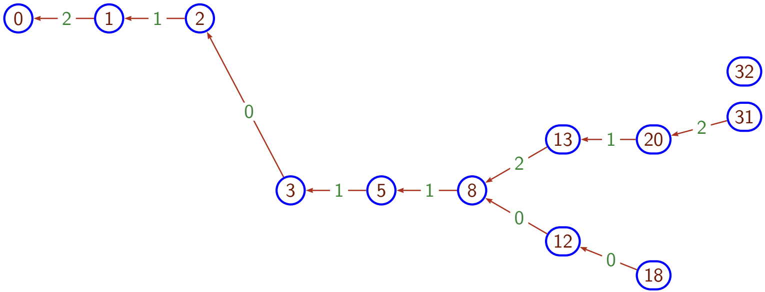

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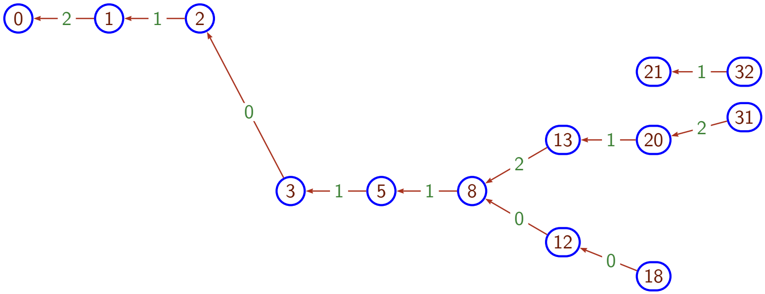

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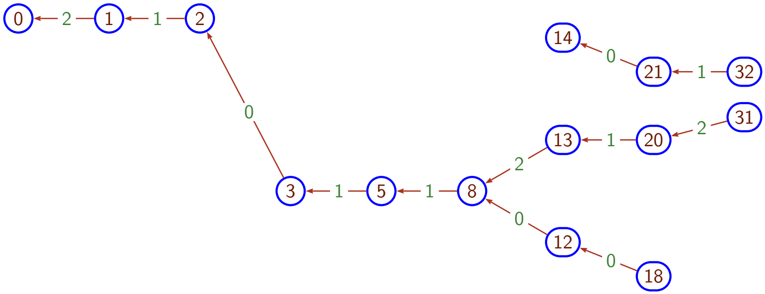

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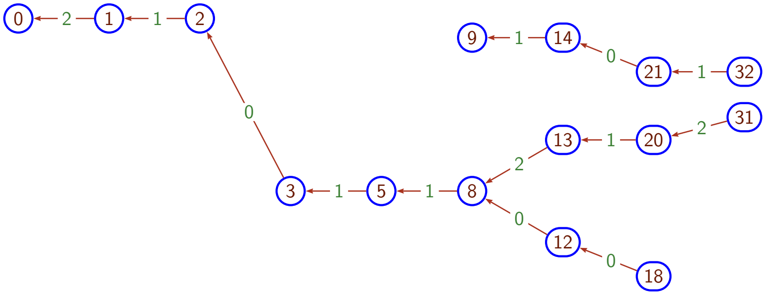

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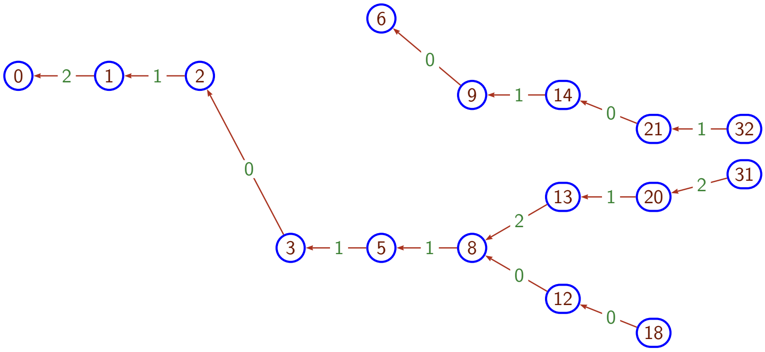

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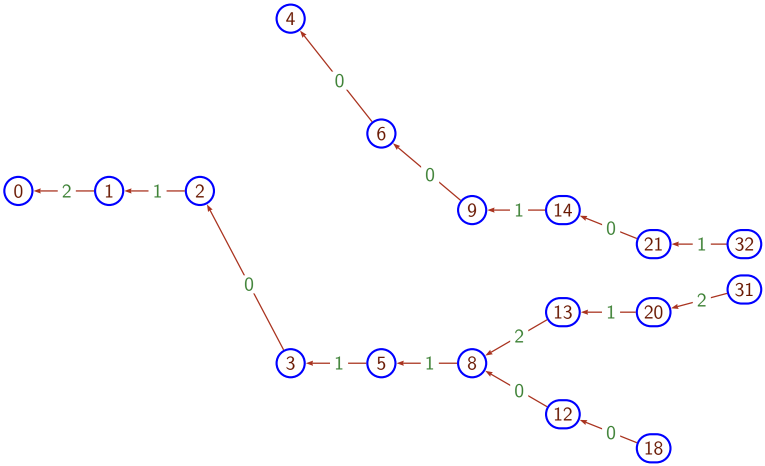

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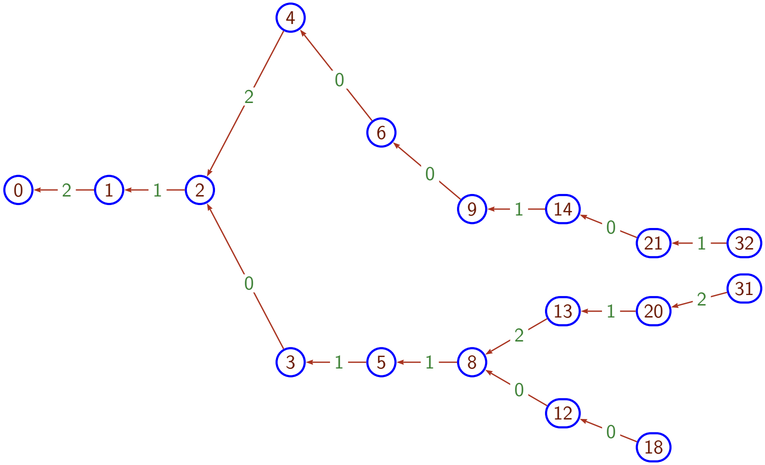

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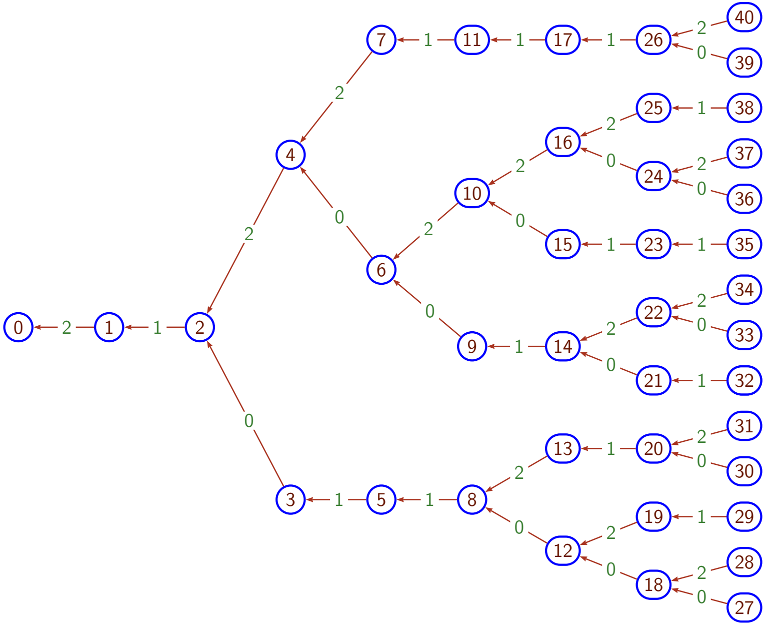
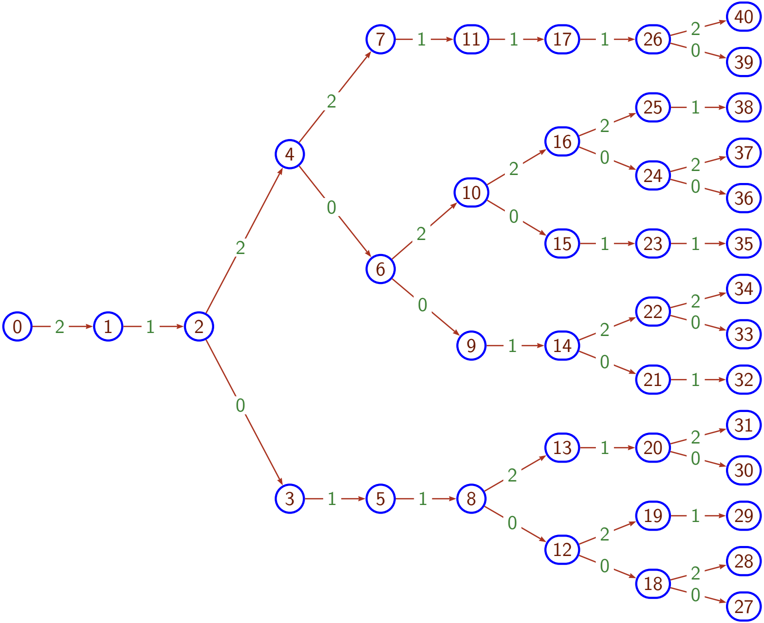


The language $L_{\frac{3}{2}}$



The evaluation function and the set of values $V_{\frac{p}{q}}$

Evaluation function:

$$\pi : \quad A_p^* \longrightarrow \mathbb{Q}$$
$$\pi(a_n \cdots a_1 a_0) = \sum_{i=0}^{n} \frac{a_i}{q} \left(\frac{p}{q}\right)^i$$

- $\pi(\langle n \rangle) = n$
- $\pi(0^* u) = \pi(u)$
- $\langle \pi(u) \rangle = u$ if u does not start with a 0 and $\pi(u)$ is an integer.

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$$V_{\frac{p}{q}} = Im(\pi) = \pi(A_p^*)$$

Outline

Properties of $L_{\frac{p}{q}}$

$$\langle m \rangle_{\frac{p}{q}} = \langle n \rangle_{\frac{p}{q}} . a$$

where (n, a) is the Euclidean division of $q \times m$ by p.

Properties of $L_{\frac{p}{q}}$

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- $L_{\frac{p}{q}}$ is prefix-closed.
- $L_{\frac{p}{q}}$ is (right-)extendable.

Properties of $L_{\frac{p}{q}}$

$$n \xrightarrow{a} m \qquad \text{iff} \qquad pn + a = qm$$

$$n \equiv n' \; [q] \quad \Longrightarrow \quad \left| \begin{array}{l} n \xrightarrow{a} \text{m} \\ n' \xrightarrow{a} \text{m'} \end{array} \right.$$

for some $m, m' \in \mathbb{N}$
and $a \in A_p$

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Theorem (Akiyama Frougny Sakarovitch, 2008)

$L_{\frac{p}{q}}$ is not a rational language.

The Bounded Left Iteration Property (BLIP)

Definition

a language L is BLIP
$\forall u\ v,\quad \exists$ only finitely indices i
such that uv^i is the prefix of a word of L.

Example : the prefixes of an infinite aperiodic word.

The Bounded Left Iteration Property (BLIP)

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Intuition 1

- L does not contain any infinite rational language.
  [IRS : Greibach 1975]
- L is “hard” to extend to a rational language.

The Bounded Left Iteration Property (BLIP)

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Intuition 2

- The topological closure of L contains **only** aperiodic word.

(Every branch of the tree-representation of L is labelled by an aperiodic word.)

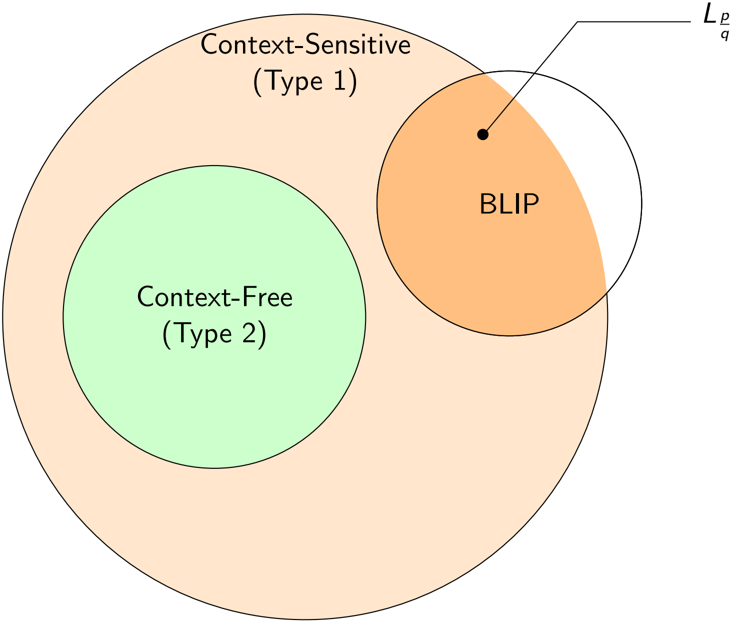
BLIP is a very robust property

- Every finite language is BLIP.

- A **finite union** of BLIP languages is BLIP.
- Any **intersection** of BLIP languages is BLIP.
- Every **sub-language** of a BLIP language is BLIP.
- The **concatenation** of two BLIP languages is BLIP.

- The **prefix closure** of a BLIP language is BLIP.
- The **inverse image by transducer** of a BLIP language is BLIP.

BLIP within formal language theory



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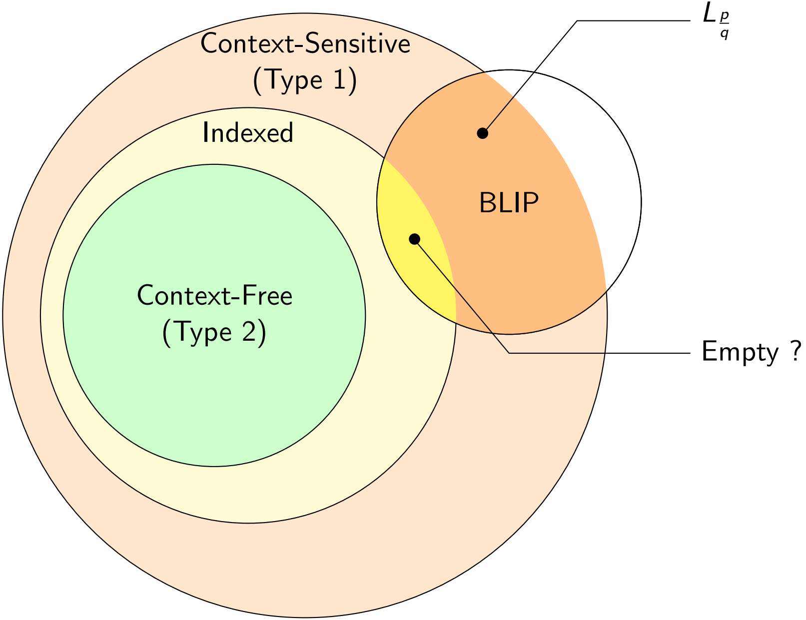

$L_{\frac{p}{q}}$ is a BLIP language

Proposition [AFS'08]

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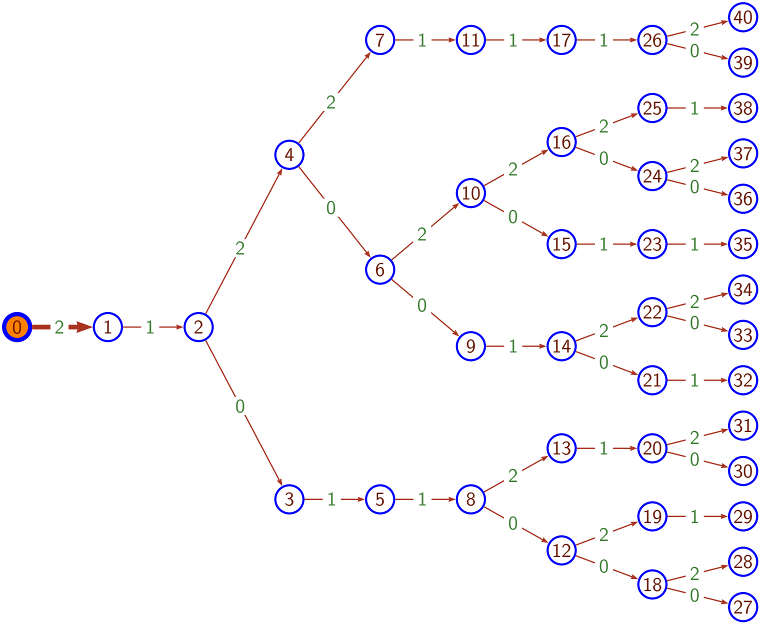
- $\implies \forall i \in \mathbb{N}, \quad u.\, v^i \in L_{\frac{p}{q}}$ and $uv.\, v^i \in L_{\frac{p}{q}}$.
- $\implies \forall i \in \mathbb{N}, \quad \pi(u) \equiv \pi(uv)\ [q^{|v| \times i}]$.
- A contradiction.

$L_{\frac{p}{q}}$ has a strong “transversal” regularity



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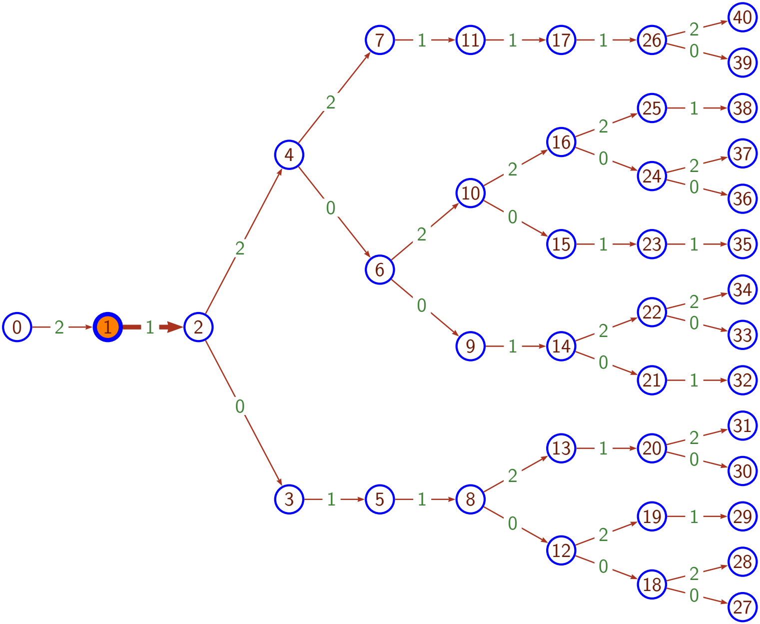

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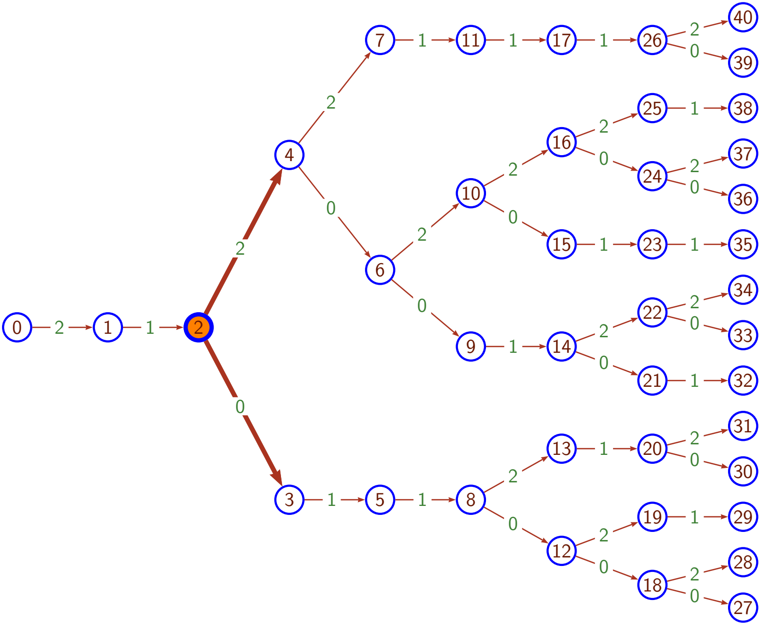

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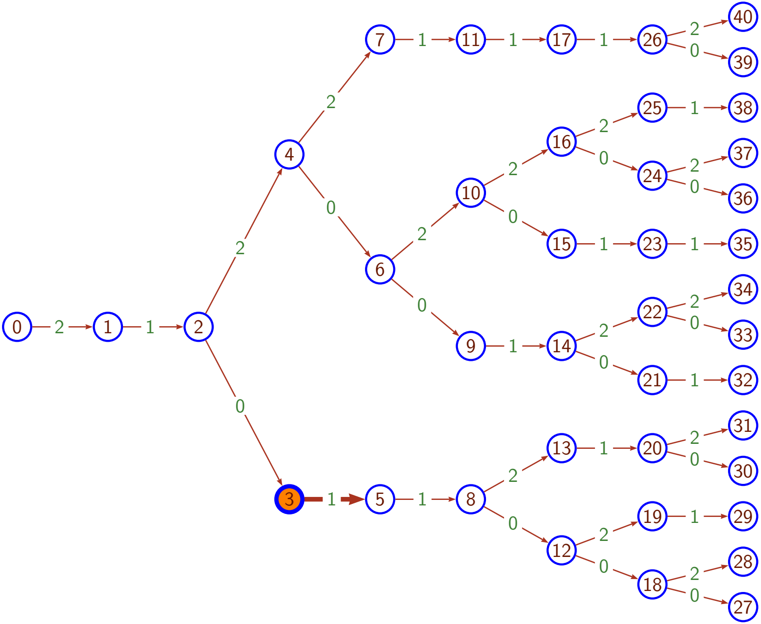

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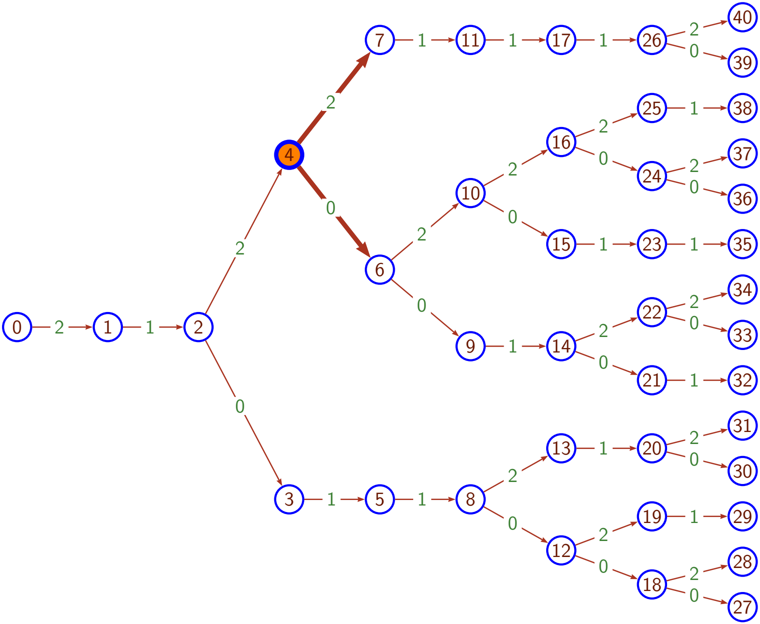

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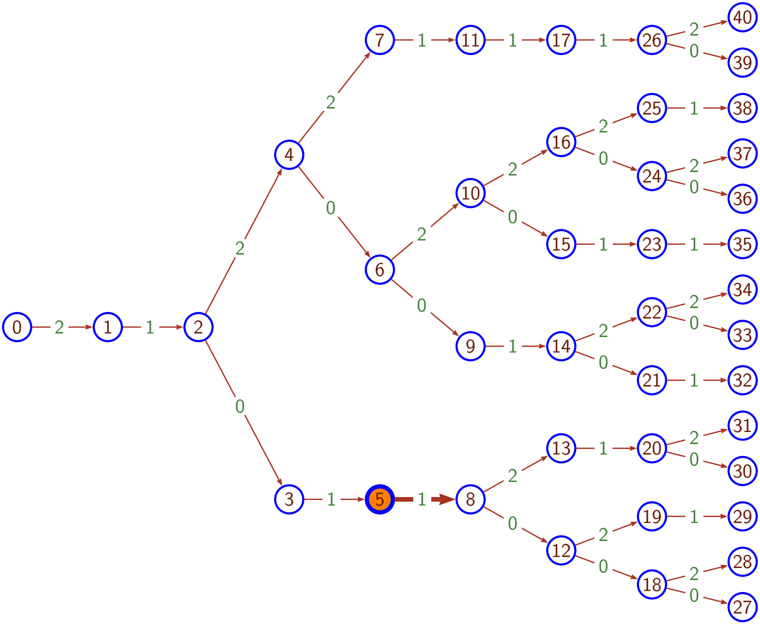

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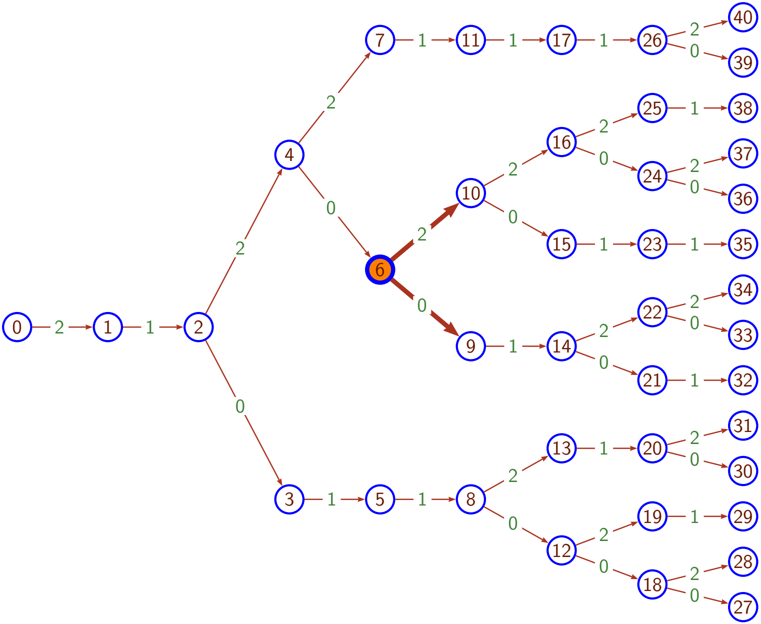

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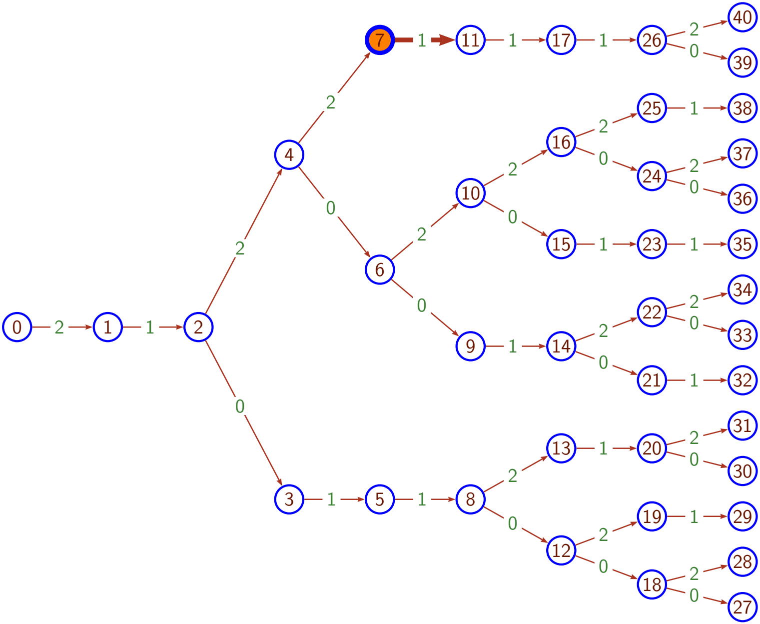

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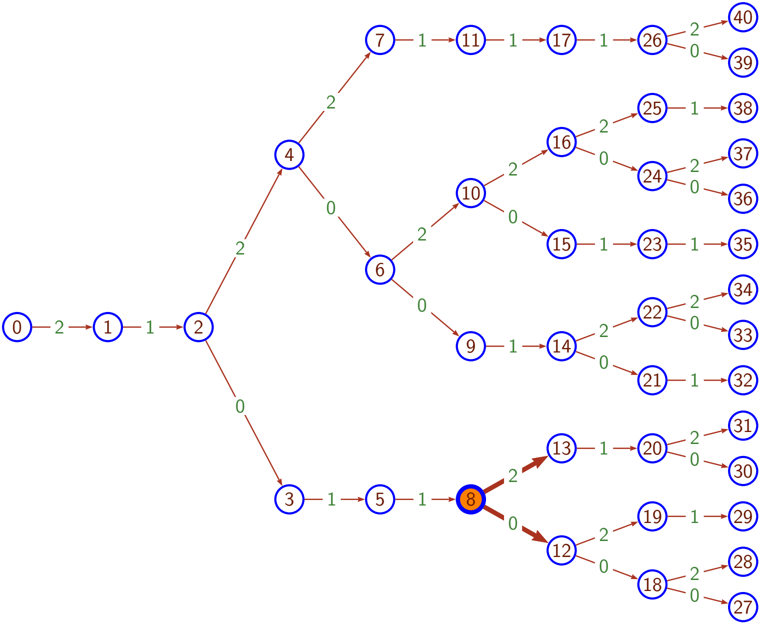

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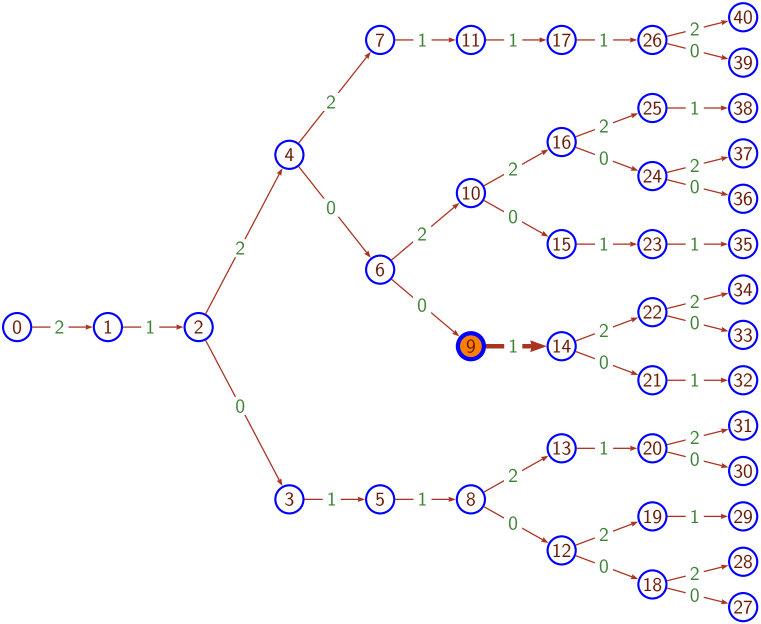

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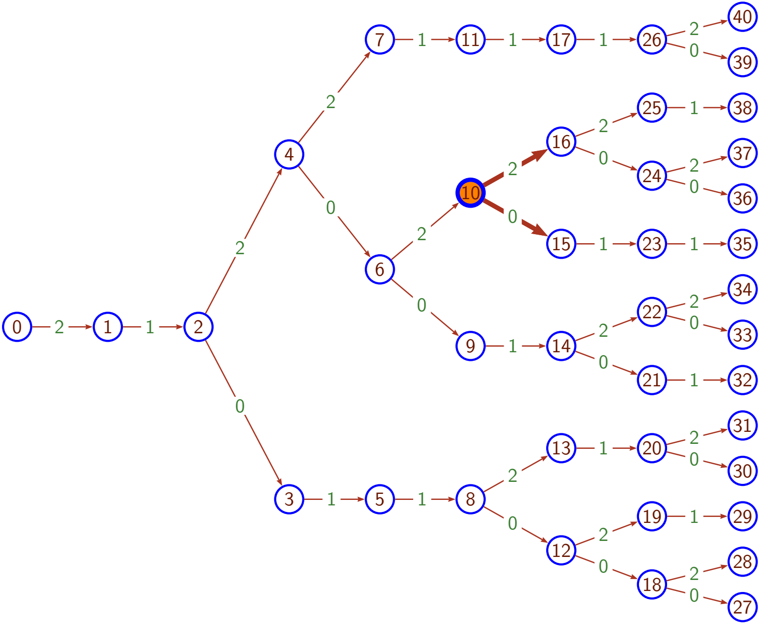

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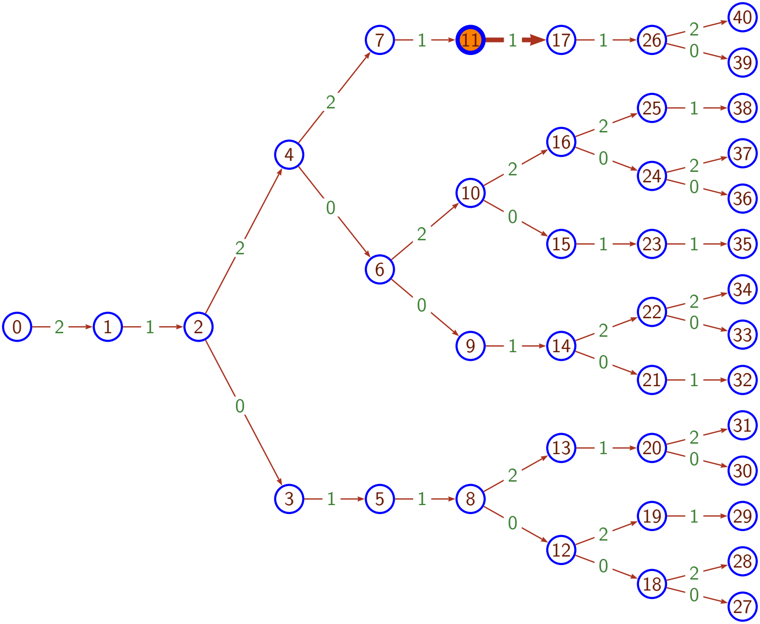

Outline

Properties of $V_{\frac{p}{q}}$

$$\pi(a_n \cdots a_1 a_0) = \sum_{i=0}^{n} \frac{a_i}{q} \left(\frac{p}{q}\right)^i$$

$$V_{\frac{p}{q}} = \pi(A_p^*)$$

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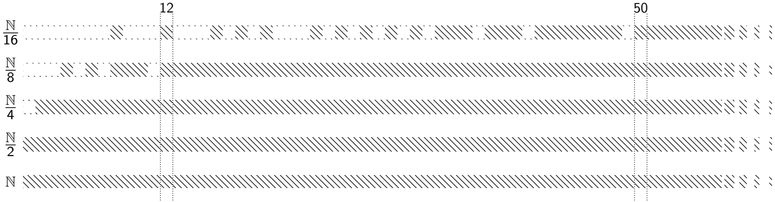
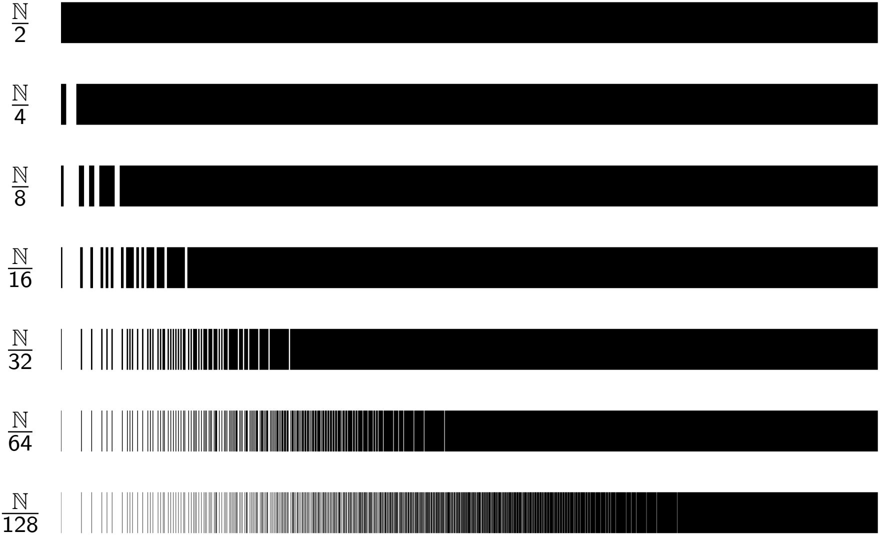
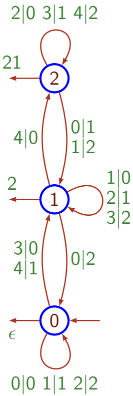


Figure: $V_{\frac{3}{2}}$, the value set in base $\frac{3}{2}$

Successive refinement of $V_{\frac{3}{2}}$



$V_{\frac{p}{q}}$ is a monoid ...and the addition is rational

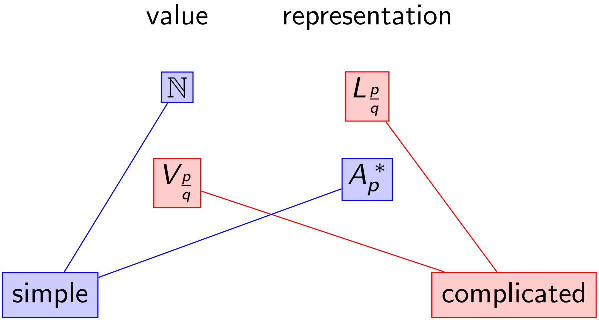


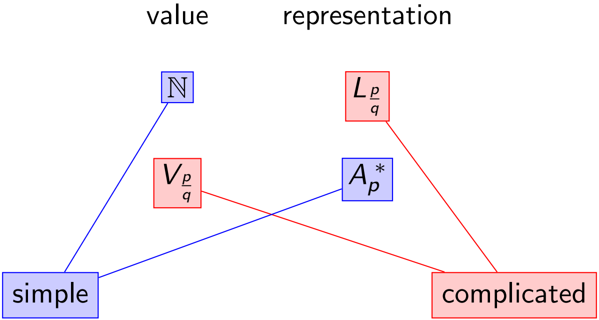
$$s \xrightarrow{a|b} t \quad \Longleftrightarrow \quad qs + a = pt + b$$

Problem

value	representation
$\mathbb{N}$	$L_{\frac{p}{q}}$
$V_{\frac{p}{q}}$	A_p^*

Problem


value
representation
$\mathbb{N}$
$L_{\frac{p}{q}}$
$V_{\frac{p}{q}}$
A_p^*
simple
complicated



Is there an object simple from both perspectives:

- value (finitely generated monoid);
- representation (rational language).

Theorem (Marsault Sakarovitch, 2013)

M: finitely generated monoid $(\subseteq V_{\frac{p}{q}})$
$\implies \langle M \rangle_{\frac{p}{q}}$ is a BLIP language.

Structure of a finitely generated submonoid of $V_{\frac{p}{q}}$

Proposition

M: finitely generated monoid $(\subseteq V_{\frac{p}{q}})$

$$M \quad \subseteq \quad \bigcup_{i \in I} (\mathbb{N} + x_i) \qquad \text{with } I \text{ finite and } x_i\text{'s} \in V_{\frac{p}{q}}.$$

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Proof.

- $g_1, g_2 \cdots g_n$: generators (of the form $\frac{n}{q^j}$, where $n, j \in N$);
- q^k : GCD of their denominator;

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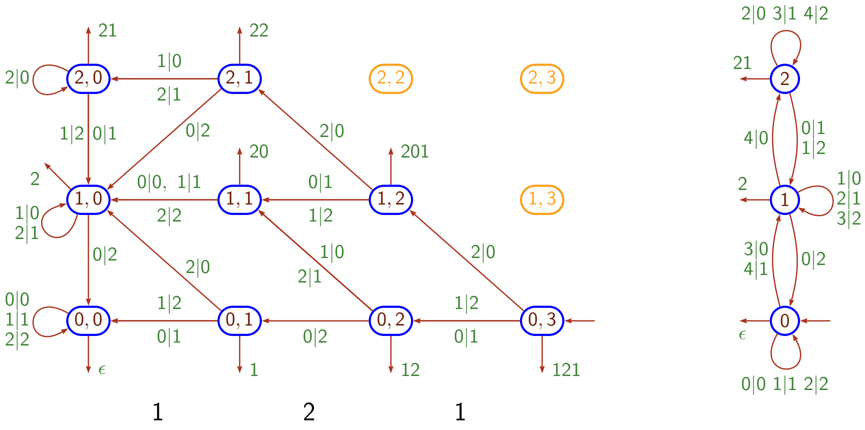
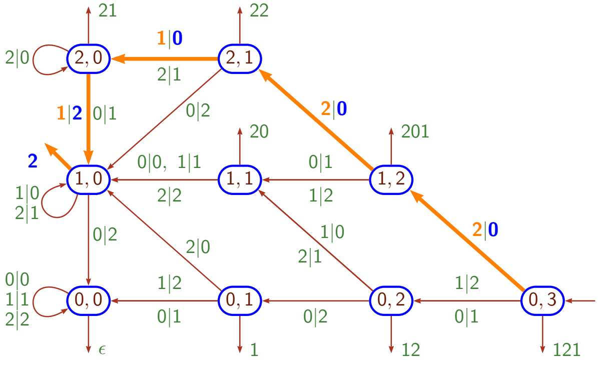
- q^k : GCD of their denominator;
- $x \in M \quad \Longrightarrow \quad x = (m + \frac{i}{q^k})$ for some $m \in \mathbb{N}$ and $i < q^k$.

Outline

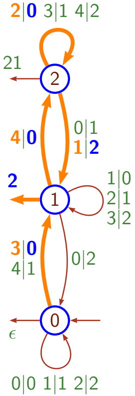
Incrementer by 3.125 (or "121") in base $\frac{3}{2}$



constant addition by 3.125 (or "121") in base $\frac{3}{2}$



	1	2	1
1	1	2	2
22	0	0	0



1 2 4 3

2 2 0 0 0

Reduction of main Statement (to Theorem B)

Theorem B

$L \subseteq A_p^*, \quad x \in V_{\frac{p}{q}}$

L is not BLIP $\Longrightarrow (L \oplus x)$ is not BLIP.

Notation

$(L \oplus x) = \langle S + x \rangle$, where $L = \langle S \rangle$.

Reduction of main Statement (to Theorem B)

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Theorem B $\Longrightarrow$ for all $y \in V_{\frac{p}{q}}, \quad (L_{\frac{p}{q}} \oplus y)$ is BLIP.

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- Ab absurdo, let us assume that $(L_{\frac{p}{q}} \oplus y)$ is not BLIP,

  $\Longrightarrow$ for all x, $(L_{\frac{p}{q}} \oplus y \oplus x)$ is not BLIP

  [Theorem B with L $= (L_{\frac{p}{q}} \oplus y)$].

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  [Theorem B with L $= (L_{\frac{p}{q}} \oplus y)$].

- We know that $\exists x \in V_{\frac{p}{q}}$, $(x + y) \in \mathbb{N}$

  $\Longrightarrow$ $(\mathbb{N} + y + x) \subseteq \mathbb{N}$

  $\Longrightarrow$ $(L_{\frac{p}{q}} \oplus x \oplus y)$ is BLIP.

L is not BLIP
$\implies \exists u, v$ and $\{w_i\}_i$, $uv^i w_i \in L$ for all i in an infinite set $\mathcal{I}$.

L is not BLIP
$\Longrightarrow$ $\exists u, v$ and $\{w_i\}_i$, $uv^i w_i \in L$ for all i in an infinite set $\mathcal{I}$.

WLOG

- $|w_i|$ arbitrarily large;
- all w_i reach the same state s of the incrementer by x;
- s is stable by every letter of v .

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- $w_i^{(\text{new})} = v\, w_i^{(\text{old})}$
- $\mathcal{I}^{(\text{new})} = \mathcal{I}^{(\text{old})} - 1 = \{i \mid (i > 0) \wedge (i+1) \in \mathcal{I}\}$

$\implies$ every $w_i^{(new)} > |v|$

Proof of: L is not BLIP $\Longrightarrow$ $L \oplus x$ is not BLIP

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- The incrementer has a finite number of states.
- The set $\mathcal{I}$ is infinite.

$\Longrightarrow$ There is a state s reached by infinitely many w_i's.

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Back to the Incrementer...

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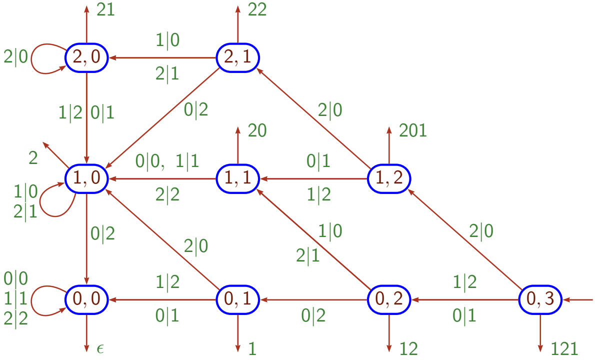

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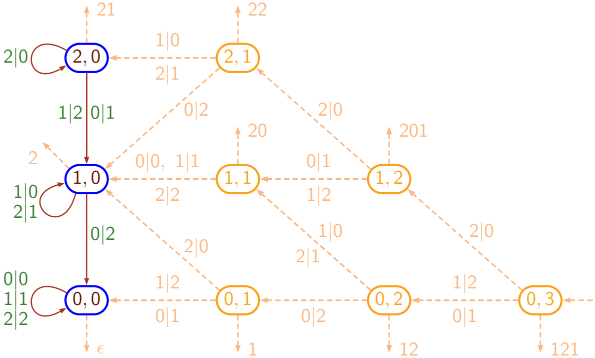

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$\implies (L \oplus x)$ is not BLIP

M finitely generated submonoid of $V_{\frac{p}{q}}$
$\implies$ $(M, +)$ is NOT an automatic structure.

Conclusion and future work

M finitely generated submonoid of $V_{\frac{p}{q}}$
$\Longrightarrow$ $(M, +)$ is NOT an automatic structure.

Conjecture

M additive submonoid $\mathbb{N} \subseteq M$ and $\langle M \rangle$ is rational.
$\langle M \rangle = X.A_p^*$ where $X = L_{\frac{p}{q}} \cap A_p^{\leq n}$