Ultimate periodicity of b-recognisable sets: a quasilinear procedure

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Séminaire, Liège
2014-01-15

Outline

Integer base

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$S \subseteq \mathbb{N}$ is b-rational

- automaton $\mathcal{A}$
- $L(\mathcal{A}) \overset{base\ b}{\longleftrightarrow} S$

Ultimately periodic sets

Definition: the class (UP)

An integer set S belongs to (UP) if there exists

- m: preperiod
- p: period
- R: remainder set

such that $\forall\, n > m, \quad n \in S \iff n \bmod p \in R.$

Properties

- (UP) is stable by union and intersection.
- Every finite set belongs to (UP).

State of the art

Theorem

Ultimately Periodic (UP) $\Longrightarrow$ b-rational

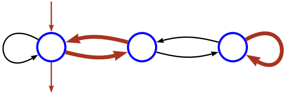

Figure: automaton accepting integers congruent to 0 modulo 3

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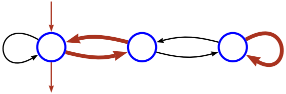

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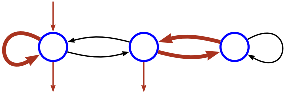

Figure: automaton accepting integers congruent to 0 or 1 modulo 3

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Fact

b-Rat $\not\Longrightarrow$ (UP)

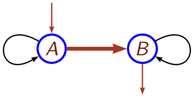

Figure: automaton accepting the powers of 2

State of the art

Theorem

Ultimately Periodic (UP) $\Longrightarrow$ b-rational

Theorem (Cobham, 1969)

$$\left.\begin{array}{l} \bullet\ S\ b_1\text{-rational} \\ \bullet\ S\ b_2\text{-rational} \\ \bullet\ b_1 \text{ and } b_2 \text{ multiplicatively independent} \end{array}\right\} \Longrightarrow S \in (\text{UP})$$

Corollary

$$(UP) = \bigcap_{b \in \mathbb{N}} b\text{-Rat}$$

Problem

ULTIMATE-PERIODICITY

PARAMETER :

- a base b

INPUT :

- a deterministic automaton $\mathcal{A}$

OUTPUT :

- Does $L(\mathcal{A}) \in$ (UP) ?

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Theorem (Honkala, 1986)

ULTIMATE-PERIODICITY is decidable.

Two orthogonal generalisations

Theorem (Leroux, 2005)

Semi-Linear($\mathbb{N}^k$) is decidable in *b-Rat*($\mathbb{N}^k$) in P-TIME.

$\rightarrow$ Quadratic complexity

$\rightarrow$ Complicated geometrical algorithm

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→ Quadratic complexity
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Other proof of Honkala's Theorem (ARS, 2009)

- '$+$' is a b-rational relation.
- The class (UP) is definable by a Presburger formula.
- Presburger arithmetic is decidable

→ Exponential complexity

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Generalisation of this method (CRS, 2012)

ULTIMATE-PERIODICITY is decidable for Pisot U-systems.

Contribution

Theorem

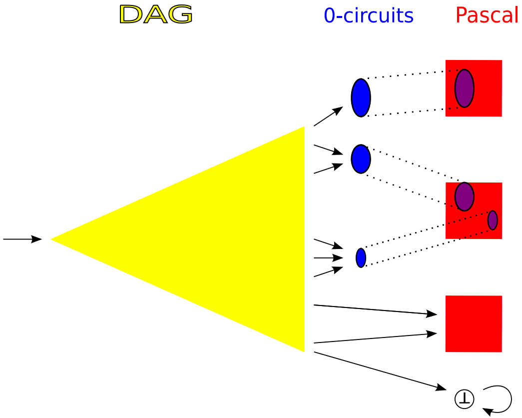
$\mathcal{A}$: a minimal deterministic automaton.
It is decidable in linear time whether $L(\mathcal{A})$ is (UP).

Corollary

ULTIMATE-PERIODICITY is solvable in quasilinear† time.

†In $O(k + n \log n)$, n being the number of states and k the number of transitions.

The UP-criterion



The UP-criterion (2)

Proposition

$\mathcal{A}$: a minimal DFA.

$\mathcal{A}$ satisfies the UP-criterion $\iff$ $L(\mathcal{A})$ is (UP).

Proposition

It is decidable in linear time whether an automaton satisfies the UP-criterion.

Outline

The Pascal automaton $\mathcal{P}_p^R$

Parameters

- (b : the base)
- p : a period, coprime with b.
- R : a set of remainders modulo p.

Expected behaviour

$$u \in {A_b}^* \text{ accepted by } \mathcal{P}_p^R \quad \Longleftrightarrow \quad \pi(u) \equiv r\ [p],\ r \in R.$$

The Pascal automaton $\mathcal{P}_p^R$ (2)

LSDF Representation yields:

- $\pi(ua) = \pi(u) + a\, b^{|u|}$
- let ψ be the smallest integer s.t. $b^{\psi} \equiv 1\ [p]$
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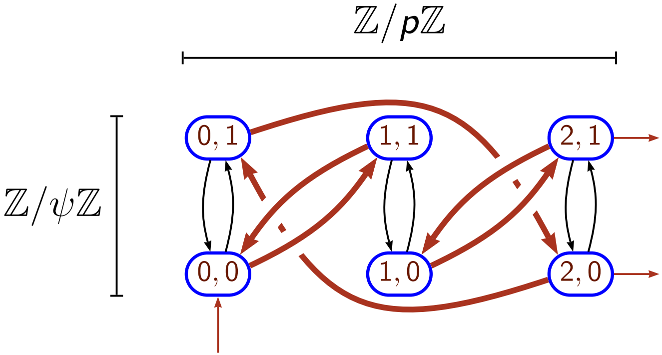
- States: $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}$

  $\pi(u) \bmod p$ → $\mathbb{Z}/p\mathbb{Z}$; $|u| \bmod \psi$ → $\mathbb{Z}/\psi\mathbb{Z}$
- Transitions: $(r, s) \xrightarrow{a} (r + ab^s, s + 1)$
- Initial state: $(0, 0)$
- Final states: $R \times \mathbb{Z}/\psi\mathbb{Z}$

Example: $\mathcal{P}_3^{\{2\}}$

- $(b = 2)$
- $p = 3$
- $\psi = 2$ (since $2^2 \equiv 1$ [3])



Structural properties

Lemma

$\mathcal{P}_p^R$ is deterministic and co-deterministic.

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Lemma (isotropism)

Changing the initial state of a Pascal automaton $\mathcal{P}_p^R$ yields a Pascal automaton $\mathcal{P}_p^S$ with the same period p but a different remainder S.

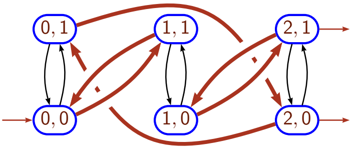
The "good" generator g

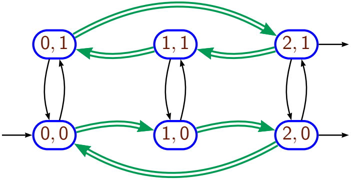
Definition

Fresh letter $g \ = \ 1\,0^{-1}$

Remark

The digit 1 is equivalent to $g0$.





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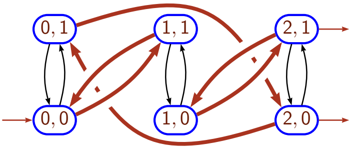
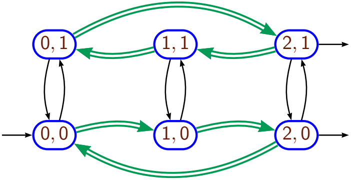
A digit d is equivalent to $\underbrace{g \cdots g}_{d \text{ times}} 0\,.$





Theorem

It is decidable in linear time whether
an automaton $\mathcal{A}$ is the quotient of a Pascal automaton.

Pascal Recognition Algorithm

Input: an automaton $\mathcal{A}$.
Output: Is $\mathcal{A}$ the quotient of a Pascal automaton?

Outline of the algorithm

- Step 1 – Simplifications : changing the alphabet
- Step 2 – Computation of the parameters (p, R, etc)
- Step 3 – Verification

Step 1 – Simplifications

Changing the alphabet of $\mathcal{A}$: $\{0,1\} \longrightarrow \{0,g\}$

1. Remove every transition labelled neither by 0 nor 1.
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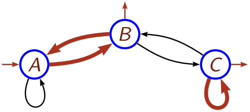

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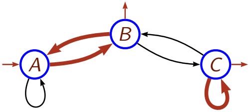

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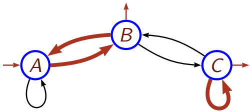

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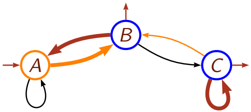

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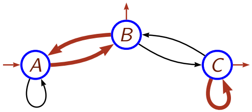

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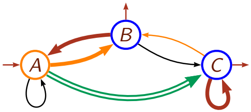

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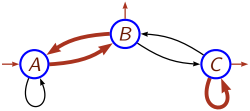

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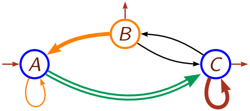

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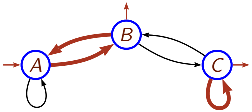

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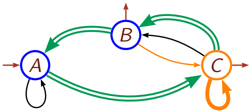

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Step 2 – Computation of the parameters

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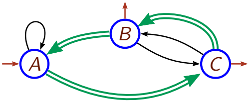
If $\mathcal{A}$ is quotient of the Pascal automaton $\mathcal{P}_p^R$; then

- The letter g induces in $\mathcal{A}$ cycles of length p.



period $p = 3$

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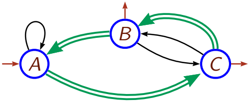

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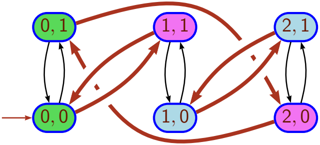

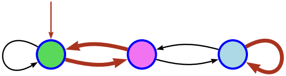

Definition: Characteristic parameter of a quotient

It is the smallest state merged with the initial state.

In the example, it is $(0, 1)$.

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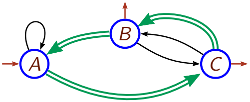
- The remainder set R contains r iff g^r is accepted by $\mathcal{A}$.
- The characteristic parameter (s, t) of the quotient is def. by

$$i \xrightarrow{g^s} \xrightarrow{0^t} i$$

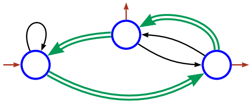


period $p = 3$
remainder set $R = \{1, 2\}$
parameter $(s, t) = (0, 1)$

Use the previous computed parameters to build *the* quotient of $\mathcal{P}_p^R$ associated with (s, t) .

- state set : $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/t\mathbb{Z}$ $3 \times 1 \;=\; 3$ states
- transitions:

$$(x, z) \xrightarrow{0} (x, z+1) \qquad \text{if } z < t-1$$
$$(x, z) \xrightarrow{0} \left(\tfrac{x-s}{b^t}, 0\right) \qquad \text{if } z = t-1$$
$$(x, z) \xrightarrow{g} (x + b^z, z)$$



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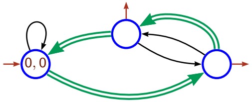

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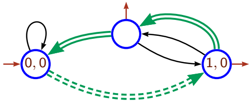

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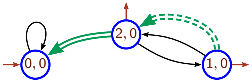

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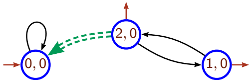

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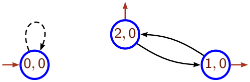

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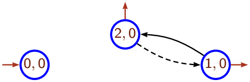

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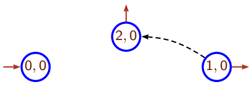

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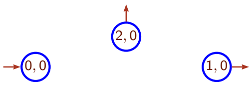
Step 3 – Verifications

Use the previous computed parameters to build *the* quotient of $\mathcal{P}_p^R$ associated with (s, t) .

- state set : $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/t\mathbb{Z}$ $3 \times 1 = 3$ states
- transitions:

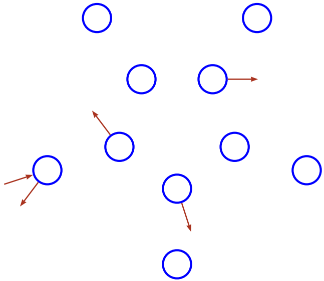
$$(x, 0) \xrightarrow{0} (2x, 0)$$

$$(x, 0) \xrightarrow{g} (x + 1, 0)$$



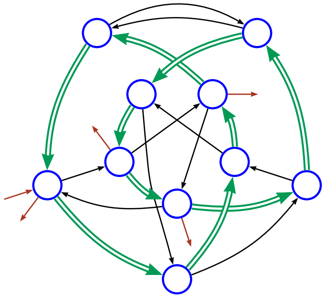
period $p = 3$
remainder set $R = \{1, 2\}$
parameter $(s, t) = (0, 1)$

Another example for base $b = 3$



Changing the alphabet of $\mathcal{A}$: $\{0, 1, 2\} \longrightarrow \{0, g\}$

1. Remove every transition labelled neither by 0 nor 1.
2. Replace every transition by 1 by one labelled by $g = 10^{-1}$.

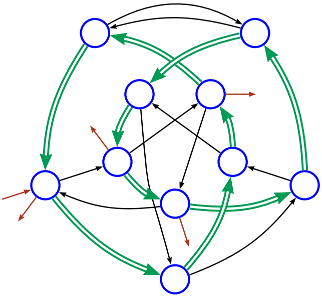


Step 2 – Computation of the parameters

Proposition

If $\mathcal{A}$ is quotient of the Pascal automaton $\mathcal{P}_p^R$; then

- The letter g induces in $\mathcal{A}$ cycles of length p.

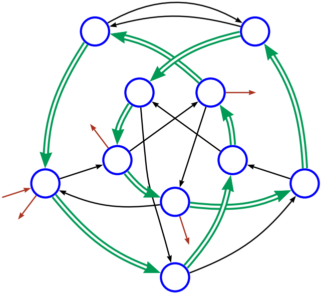


period $p = 5$

Proposition

If $\mathcal{A}$ is quotient of the Pascal automaton $\mathcal{P}_p^R$; then

- The letter g induces in $\mathcal{A}$ cycles of length p.
- The remainder set R contains r iff g^r is accepted by $\mathcal{A}$.



period $p = 5$
remainder set $R = \{0, 3\}$

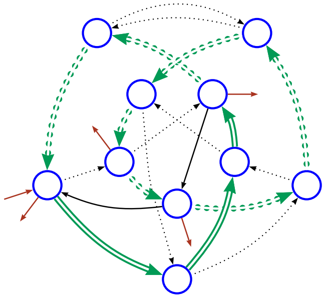
Step 2 – Computation of the parameters

Proposition

If $\mathcal{A}$ is quotient of the Pascal automaton $\mathcal{P}_p^R$; then

- The letter g induces in $\mathcal{A}$ cycles of length p.
- The remainder set R contains r iff g^r is accepted by $\mathcal{A}$.
- The characteristic parameter (s, t) of the quotient is def. by

$$i \xrightarrow{g^s} \xrightarrow{0^t} i$$



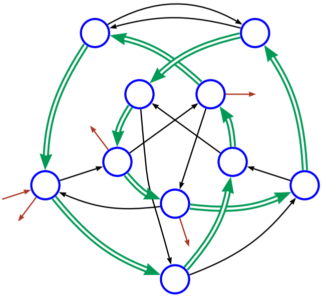
period $p = 5$
remainder set $R = \{0, 3\}$
parameter $(s, t) = (3, 2)$

Step 3 – Verifications

- state set : $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/t\mathbb{Z}$ $\quad 5 \times 2 = 10$ states
- transitions:

$$(x,z) \xrightarrow{0} (x, z+1) \qquad \text{if } z < t-1$$
$$(x,z) \xrightarrow{0} \left(\tfrac{x-s}{b^t}, 0\right) \qquad \text{if } z = t-1$$

$$(x,z) \xrightarrow{g} (x + b^z, z)$$



period $p = 5$

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parameter $(s, t) = (3, 2)$

$\frac{1}{b^t} \bmod p = \frac{1}{9} \bmod 5 = 4$

$\frac{x-s}{b^t} = 3 - x$

Step 3 – Verifications

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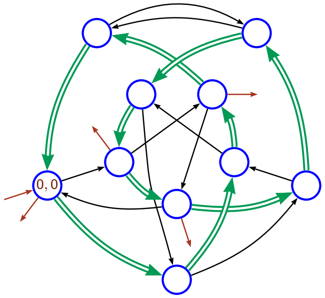

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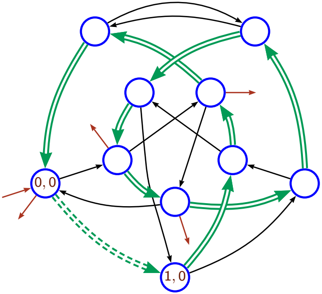

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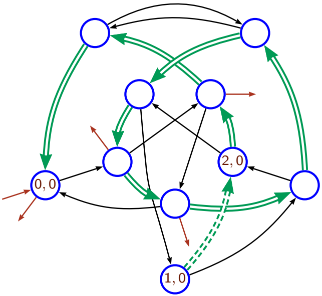

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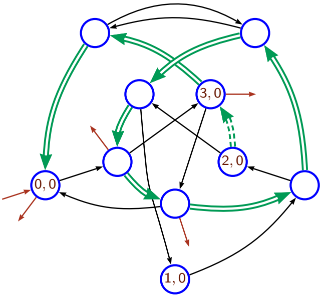

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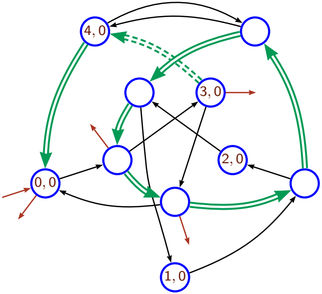

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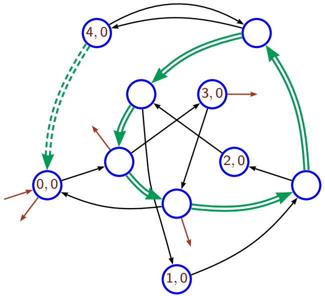

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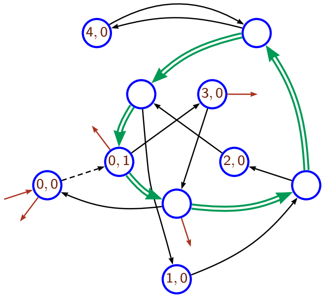

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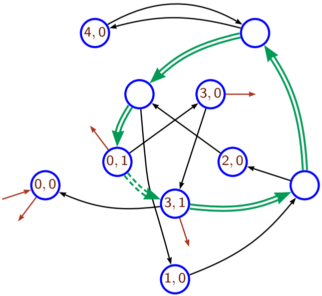

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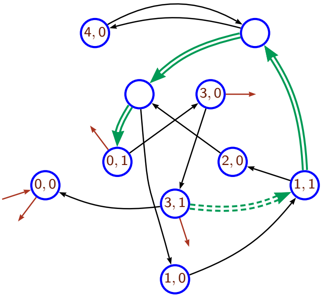

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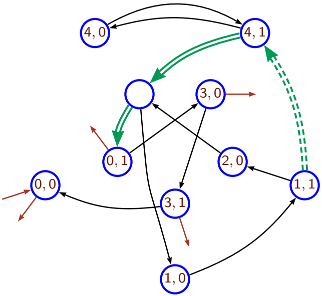

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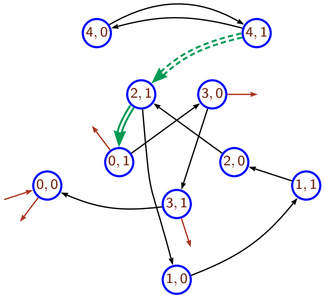

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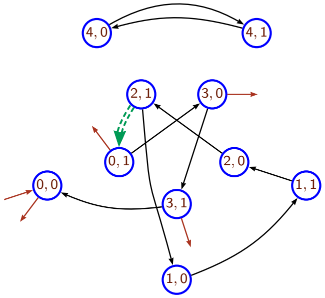

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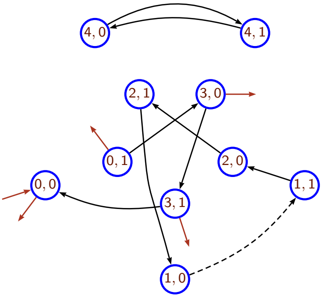

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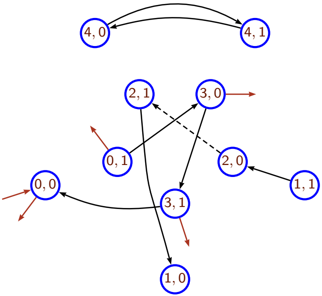

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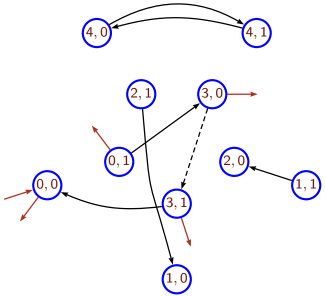

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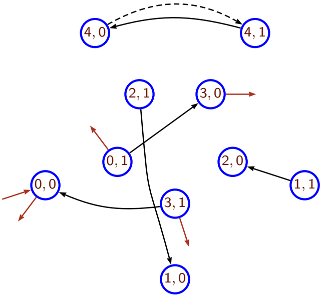

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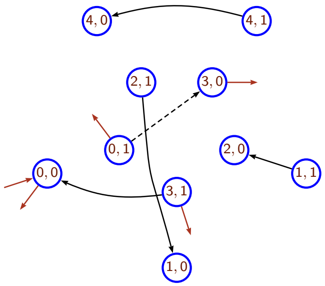

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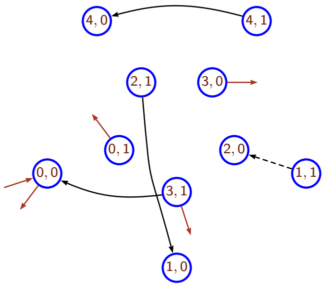

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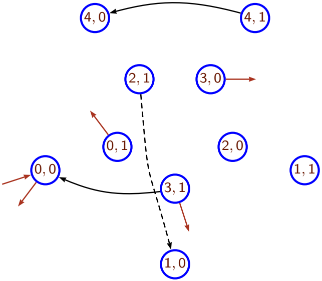

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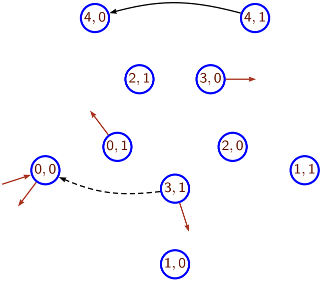

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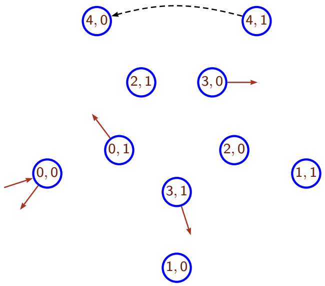

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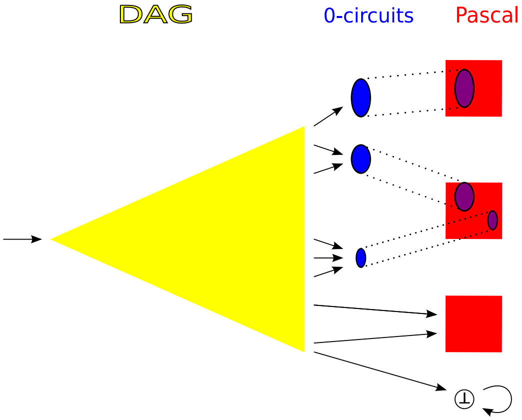
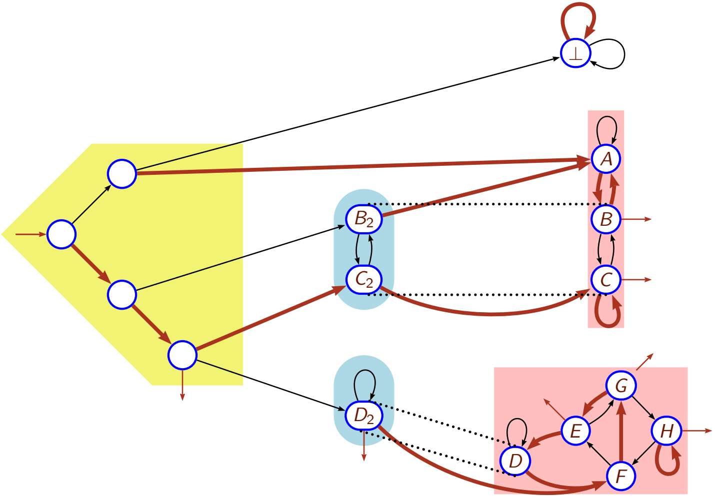
Outline



DAG
0-circuits
Pascal
⊥

An automaton satisfying the UP-criterion



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It can be decided in linear time whether
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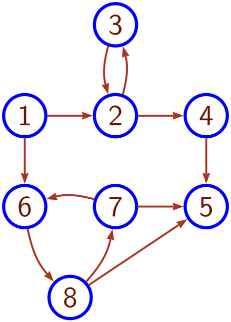
Linear Complexity (1) – Condensation



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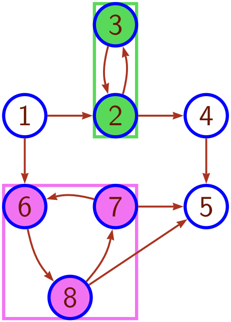

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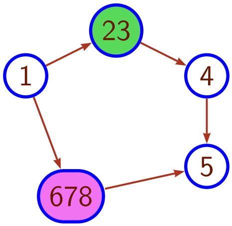
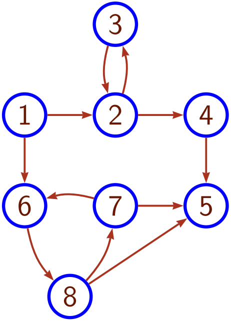
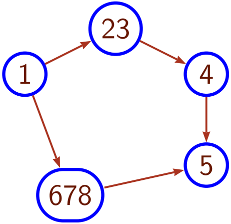


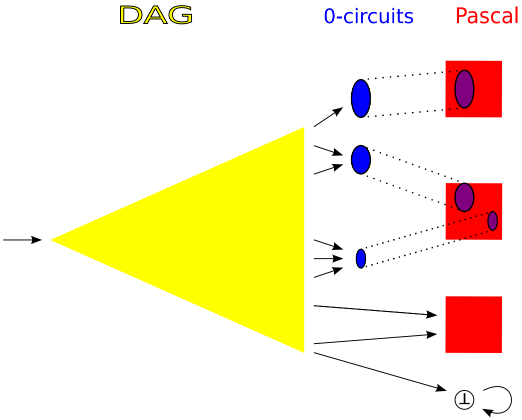




Algorithm (Tarjan, 1972)

Computing the condensation of a graph takes linear time.

Linear Complexity (2)



Verifying that an automaton satisfies the UP-criterion consists in a simple browsing of the automaton and its condensation.

Completeness

Theorem (Completeness)

$\mathcal{A}$: a *minimal* automaton.
$L(\mathcal{A})$ is (UP) $\Rightarrow$ $\mathcal{A}$ satisfies the UP-criterion.

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Completeness – $R \mod b^i$

Expected behaviour

A word u is accepted $\iff \pi(u) \mod b^i$ belongs to R

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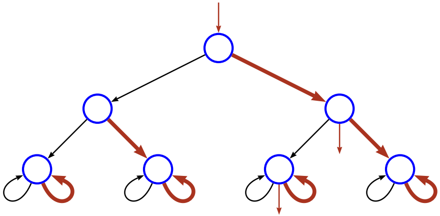

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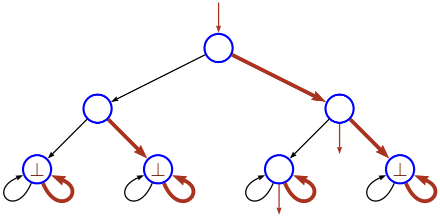

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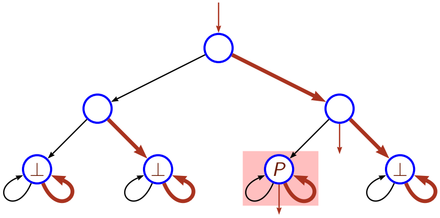

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Completeness – $r \mod (k \times b^i)$

Chinese remainder theorem

$$n \equiv r \mod (k \times b^i) \iff \begin{cases} n \equiv r_k \mod k \\ n \equiv r_d \mod b^i \end{cases}$$

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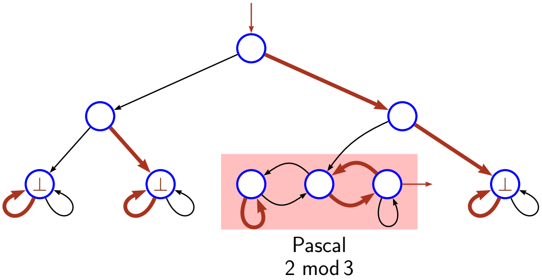
Example: Automaton accepting integers 5 mod 12

$$n \equiv 5 \bmod (3 \times 2^2) \iff \begin{cases} n \equiv 2 \bmod 3 \\ n \equiv 1 \bmod 4 \end{cases}$$



Completeness – $R \bmod (k \times b^i)$

Automaton accepting integers $0, 5, 8 \bmod 12$

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\begin{array}{lclcl}
0 \bmod (3 \times 4) & \iff & 0 \bmod 4 & \text{and} & 0 \bmod 3 \\
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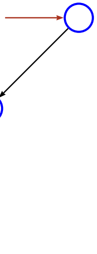

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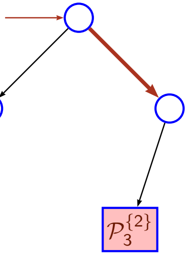

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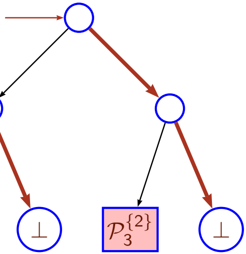
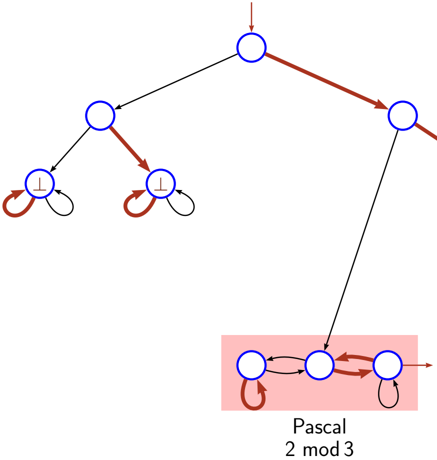


Completeness – R mod $(k \times b^i)$ and greater than m

Example

Automaton accepting integers $n > 5$ and $n \equiv 5 \bmod (3 \times 2^2)$



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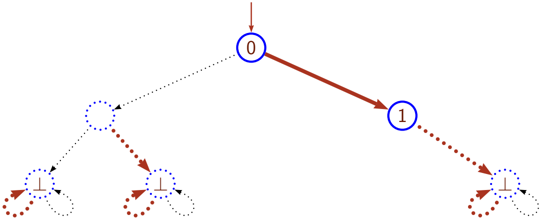

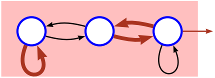

Pascal
2 mod 3

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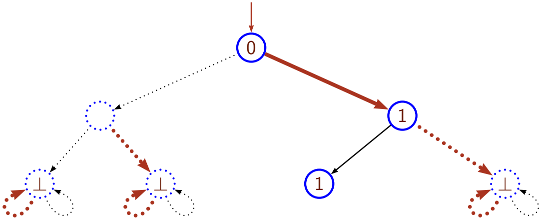

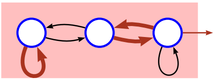

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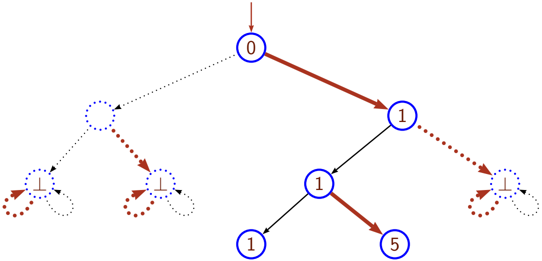

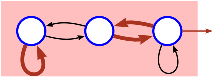

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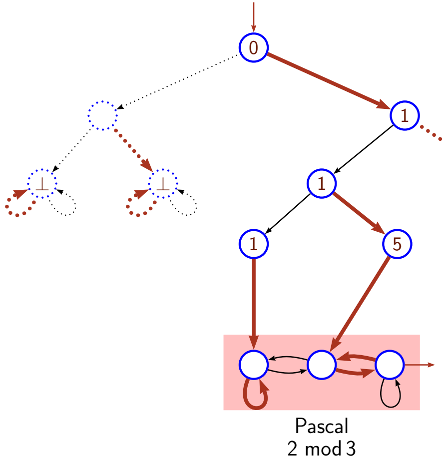

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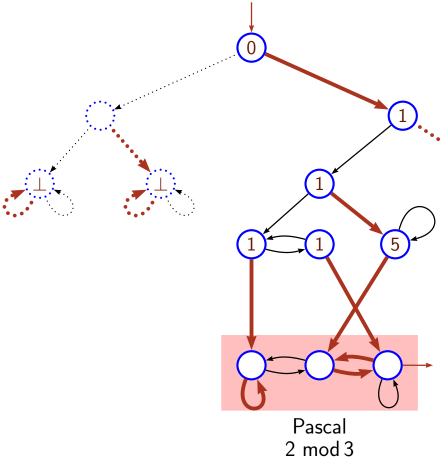
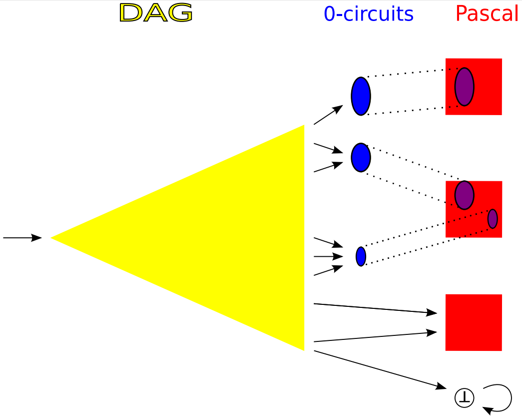


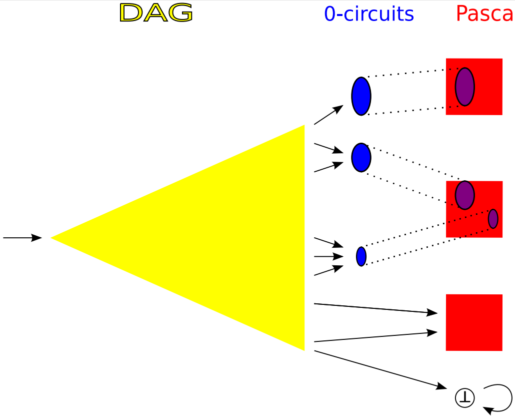
Completeness – Summary



a set of (UP) : $\{n \mid n > m \text{ and } n \equiv r\ [p] \text{ with } r \in R\}$

- preperiod
- period
- remainder set

Completeness – Summary



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- period $\Rightarrow$ Pascal's period & DAG depth.
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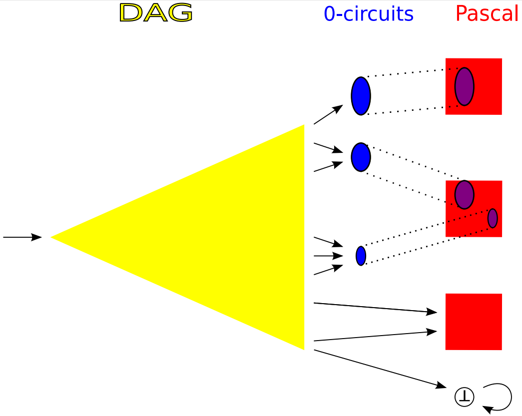

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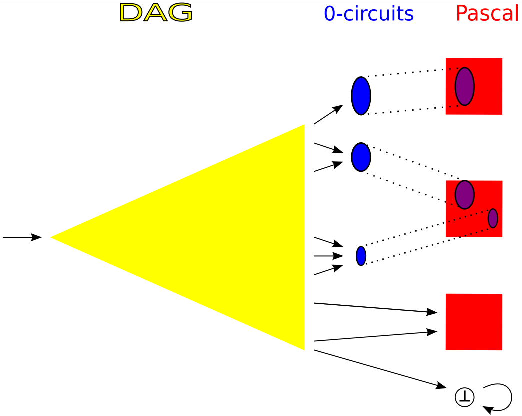

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- preperiod ⇒ 0-circuits & DAG depth.
- period ⇒ Pascal's period & DAG depth.
- remainder set ⇒ # of Pascal's & Pascal's remainder sets

Proof of Correctness on an example (1)

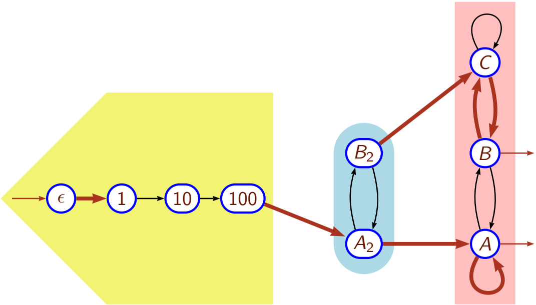
Theorem (Correctness)

$\mathcal{A}$: a *minimal* automaton.
$\mathcal{A}$ satisfies the UP-criterion $\Rightarrow L(\mathcal{A})$ is (UP).

WLOG we can assume that, in $\mathcal{A}$,

- the only final states are in the Pascal automata
  [since other final states accepts only finitely many integer];
- the DAG-part is a simple path
  [since the union of (UP) sets is a (UP) set].

Proof of Correctness on an example (2)



Lemma (isotropism)

Changing the initial state of a Pascal automaton $\mathcal{P}_p^R$ yields a Pascal automaton $\mathcal{P}_p^S$ with the same period p but a different remainder S.

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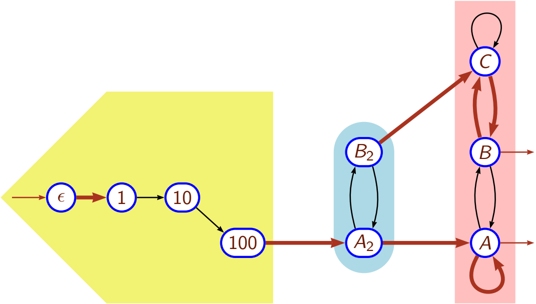

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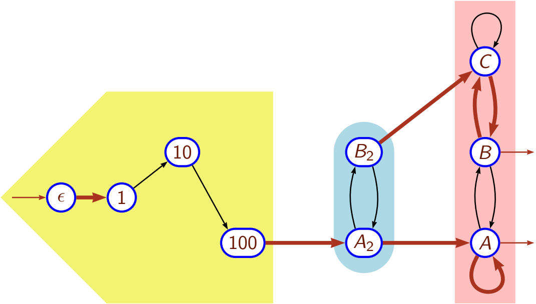

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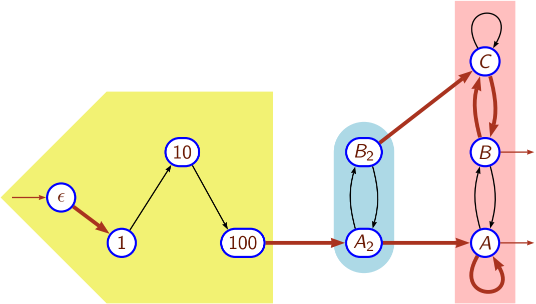

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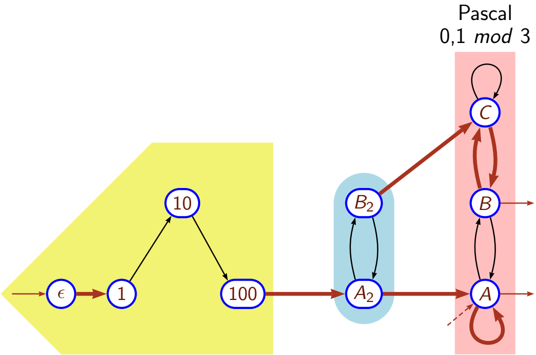

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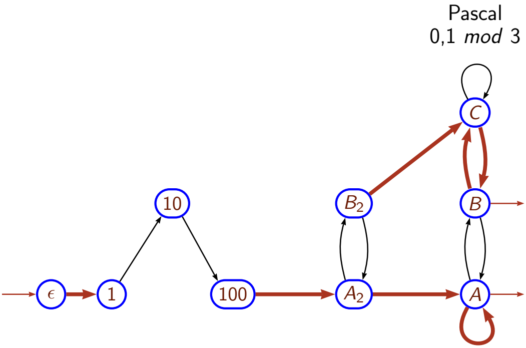
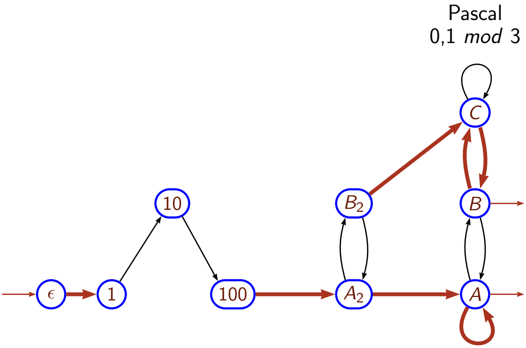


A word u is accepted if and only if

u reaches the Pascal aut. and $\pi(u) \equiv 0{,}1 \bmod 3$.

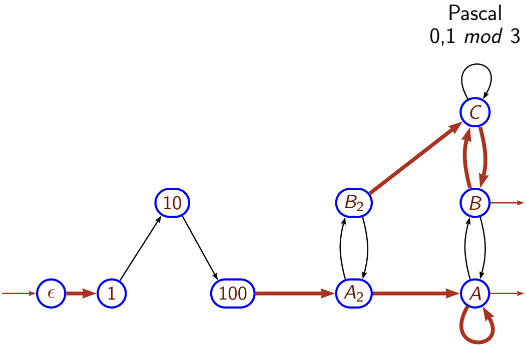
Proof of Correctness on an example (2)



A word u is accepted if and only if

$$u \text{ starts with } 10010^i1 \quad \text{and} \quad \pi(u) \equiv 0,1 \bmod 3.$$

Proof of Correctness on an example (2)



A word u is accepted if and only if

$$\underbrace{u \text{ starts with } 10010^i1}_{\iff\ u > 16 \text{ and } \pi(u) \equiv 9\ (16)} \quad \text{and} \quad \pi(u) \equiv 0{,}1 \bmod 3.$$

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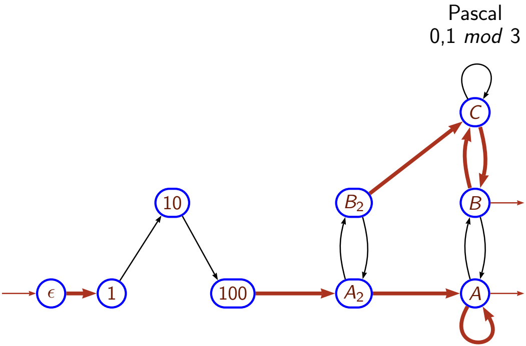

A word u is accepted if and only if

$$\pi(u) > 16 \quad \text{and} \quad \pi(u) \equiv 9 \bmod 16 \quad \text{and} \quad \pi(u) \equiv 0{,}1 \bmod 3.$$

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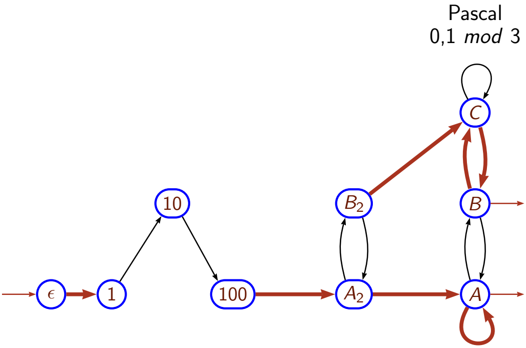

A word u is accepted if and only if

$$\pi(u) > 16 \quad \text{and} \quad \pi(u) \equiv 9, 25 \bmod 48.$$

Outline

Conclusion

- Quasilinear algorithm to decide whether a DFA is (UP)
- Structural characterisation of minimal (UP) DFA

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Future work

- Getting rid of the minimality condition
  → Work in progress...
- Getting rid of determinism condition
  → Seems unrealistic with this method.
- Generalising this method to U-Systems
  → The "*isotropism* lemma" is false in the general case
  → Yields an EXP-TIME algorithm (no better than [ASR'09])
  → In nice cases (Fibonacci, Tribonacci, etc), it may yield a P-TIME algorithm.