Rhythmic generation of infinite trees and languages

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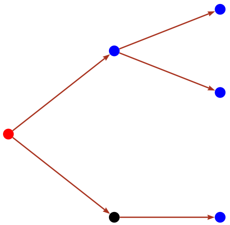
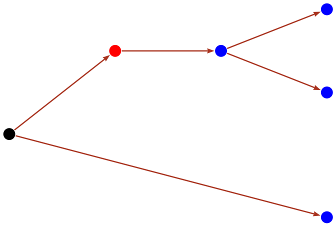
Outline

Trees are rooted (*ie.* directed outward the root)



$\neq$



Trees are embedded in the plane

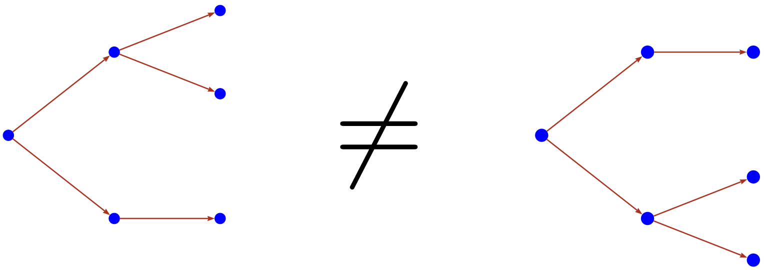
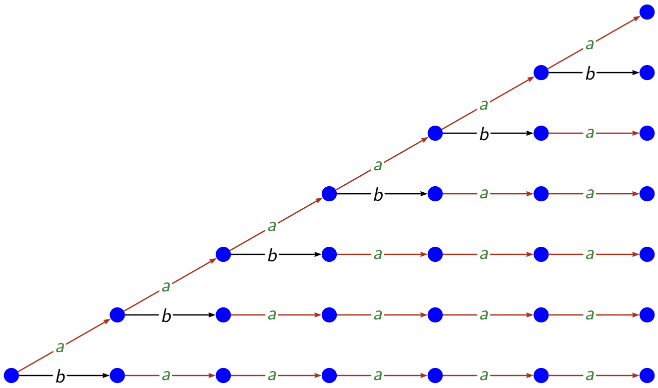


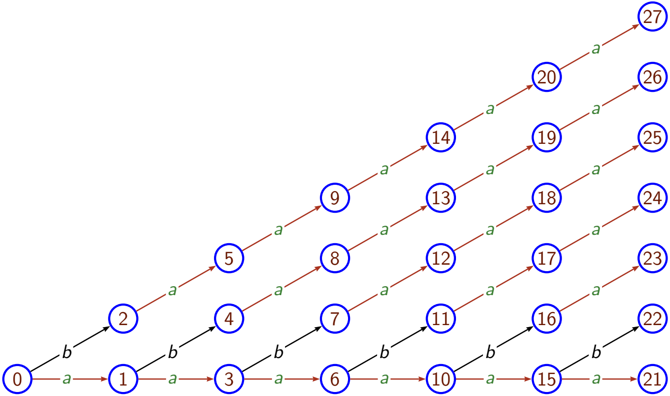
Figure: The langage $a^*ba^* + a^*$ (of the words with 0 or 1 b).



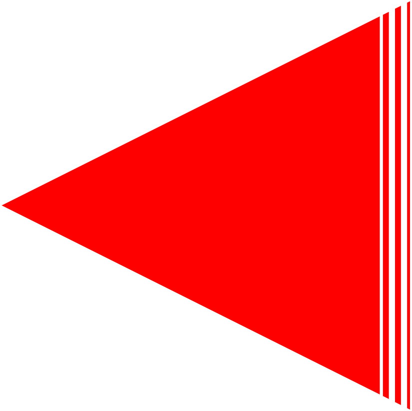
Languages are radix-ordered

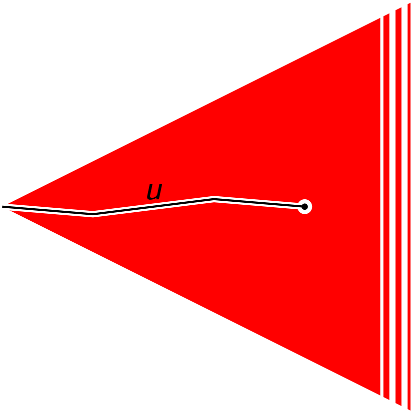
Definition (radix order $<_{\text{rad}}$)

$$u <_{\text{rad}} v \quad \text{if} \quad |u| < |v| \quad \text{or} \quad |u| = |v| \;\&\; |u| <_{\text{lex}} |v|$$



Automata are related to depth-first traversal.

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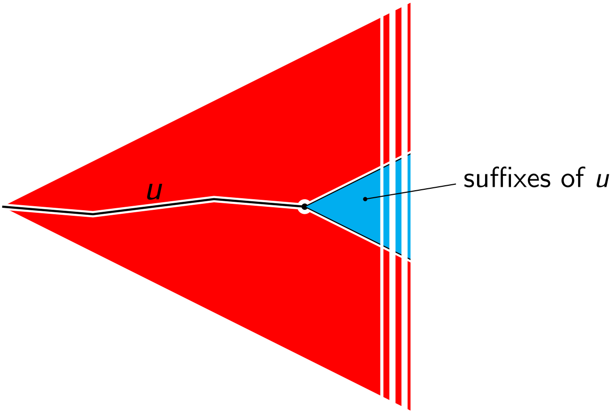

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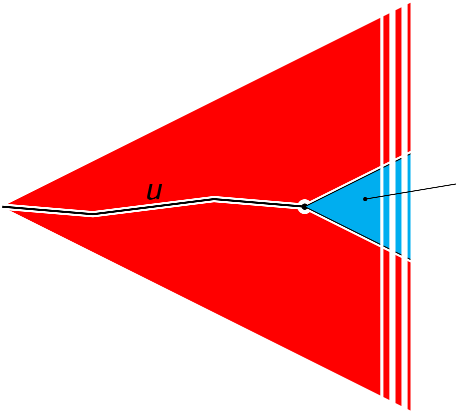


suffixes of u

- depends on a summary of u
  (state of the respective DFA)

Theorem

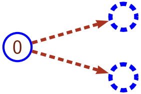
$\mathbf{r}$: *a rhythm*
(q, p): *its directing parameter*
$K_{\mathbf{r}}$: *the language generated by the rhythm* $\mathbf{r}$

- *If* $\frac{p}{q}$ *is an integer* $K_{\mathbf{r}}$ *is a rational language.*
- *If* $\frac{p}{q}$ *is not integer* $K_{\mathbf{r}}$ *is a BLIP language.*

Definition (rhythm)

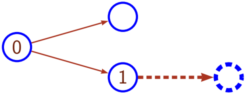
$$\mathbf{r} \;=\; (r_0,\, r_1,\, \ldots\, ,\, r_{q-1})$$

$$\mathbf{r} = (\,3\,,\,1\,,\,1\,)$$



The tree generated by $(3, 1, 1)$

$$\mathbf{r} = (3, 1, 1)$$



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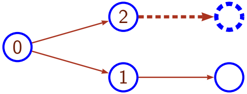

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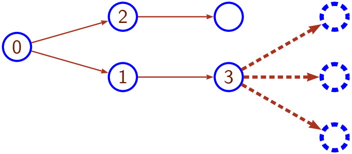

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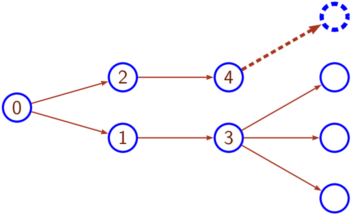

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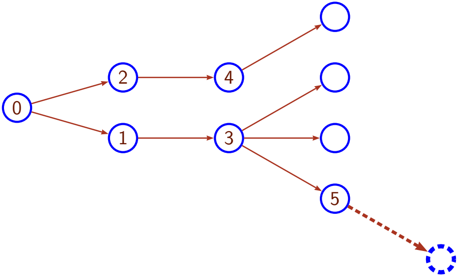

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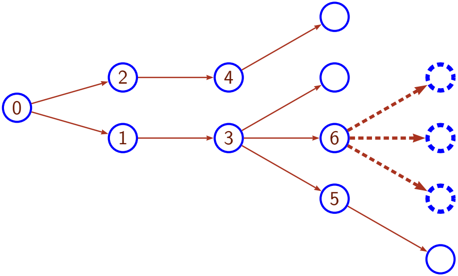

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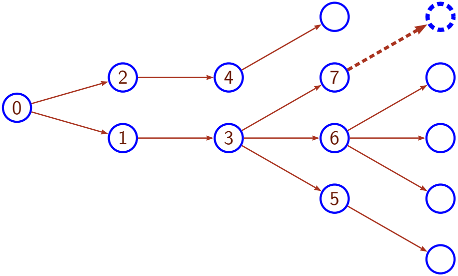

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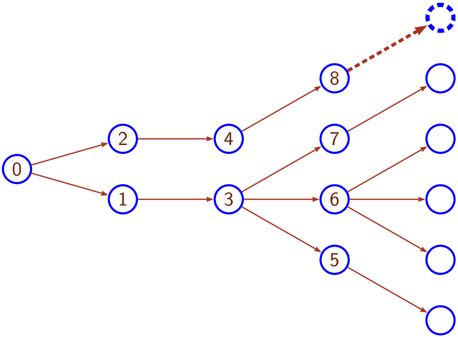

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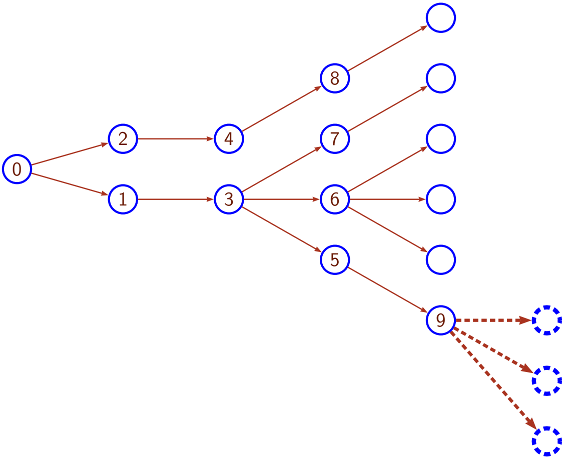

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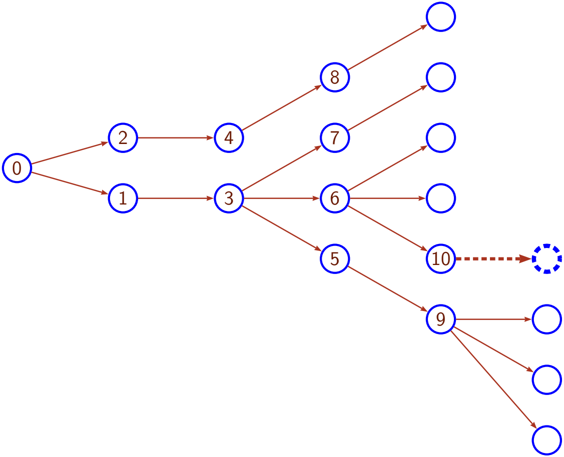

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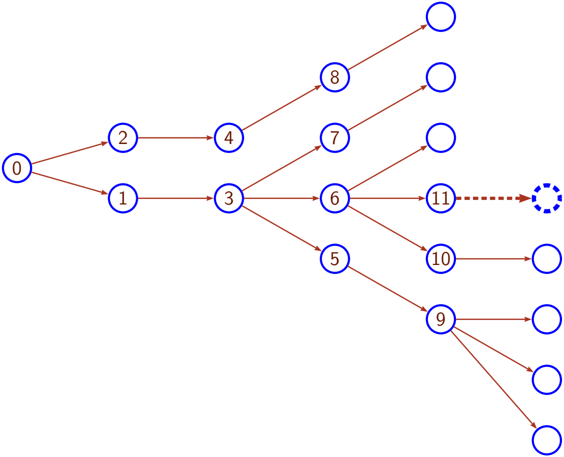

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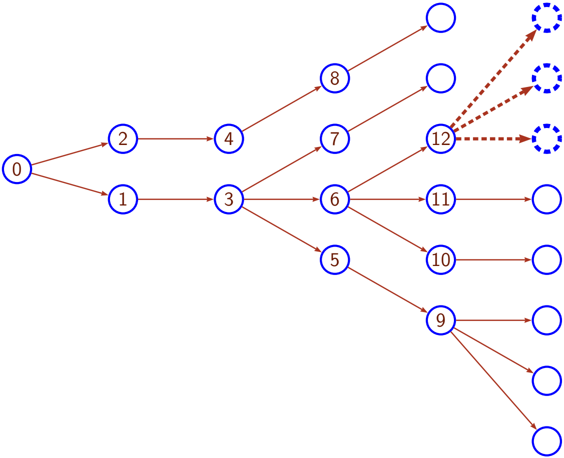

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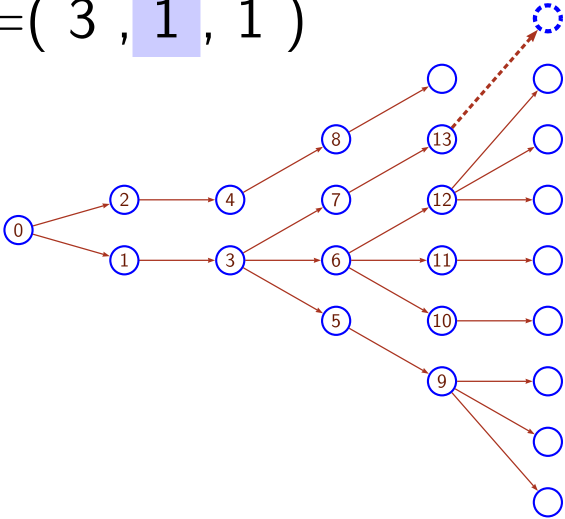

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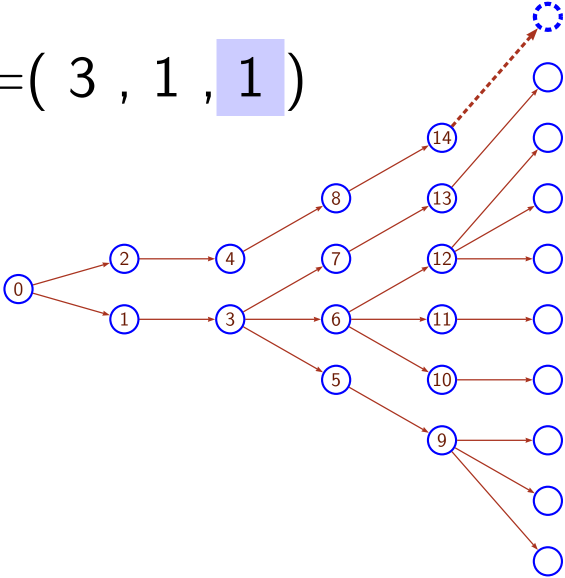


Directing parameter & Growth ratio

A rhythm $\mathbf{r} = (\ r_0\ ,\ \ r_1\ ,\ \ r_2\ , \ldots ,\ \ r_{q-1}\ )$

- Directing parameter (q, p):
  - $\mathbf{r}$ is a q-tuple;
  - $p\ =\ r_0 + r_1 + r_2 + \cdots + r_{q-1}$.
- Growth ratio: $\frac{p}{q}$
  - Intuition : $\#\{\text{nodes at depth } i\}$ is roughly $\left(\frac{p}{q}\right)^i$

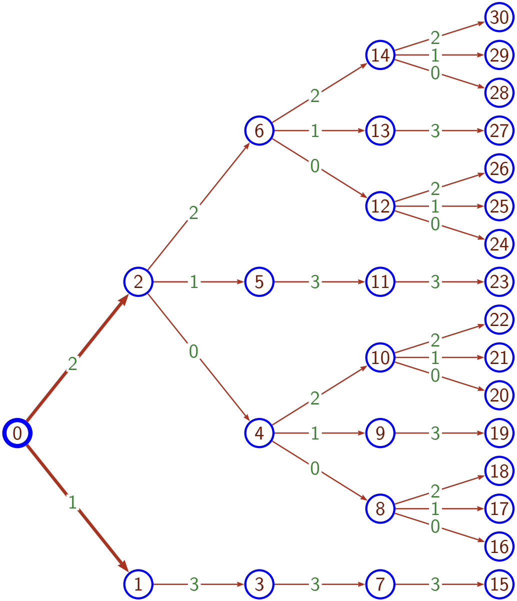
Generating languages by rhythm

Definition (Naive labelling)

The edge $n \xrightarrow[\mathcal{T}_{\mathbf{r}}]{a} m$ is labelled by $a = (m \mod p)$

Intuition: The transition labels follows $0, 1, 2, \ldots, (p-1), 0, 1 \ldots$

The language generated by (3, 1)



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$(\ 0,\ 1,\ 2\ ,\ 3\ )$

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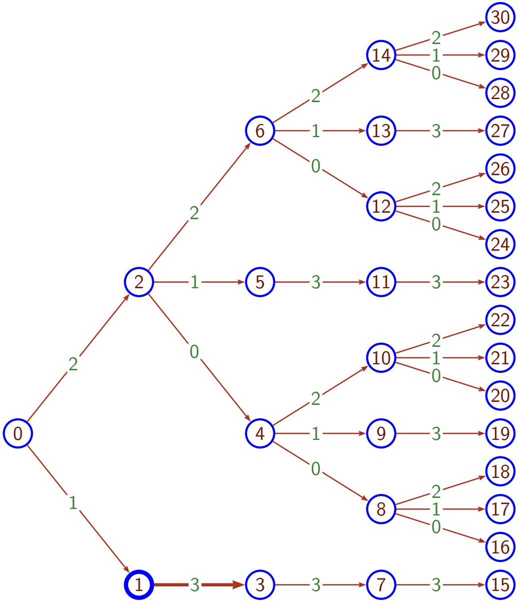

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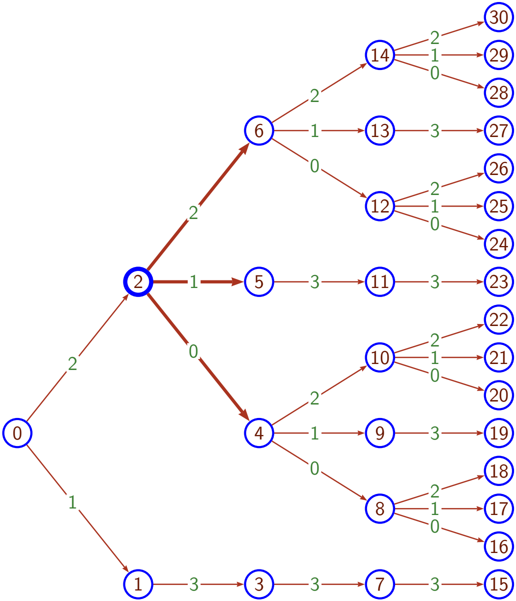

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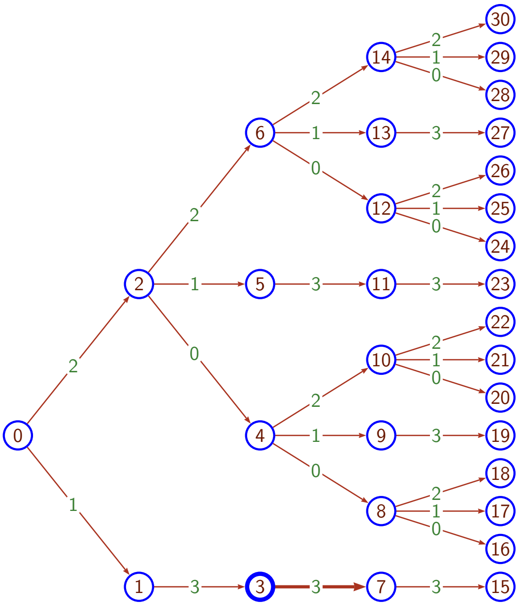

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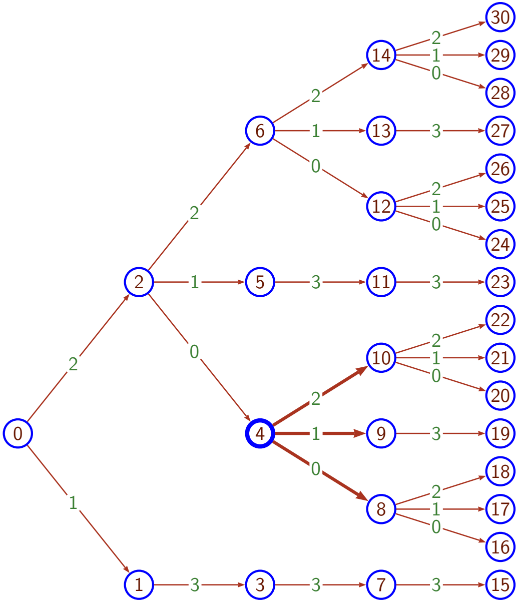

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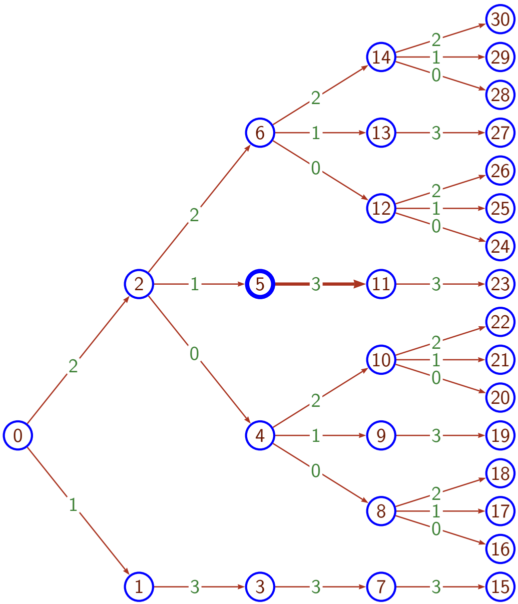

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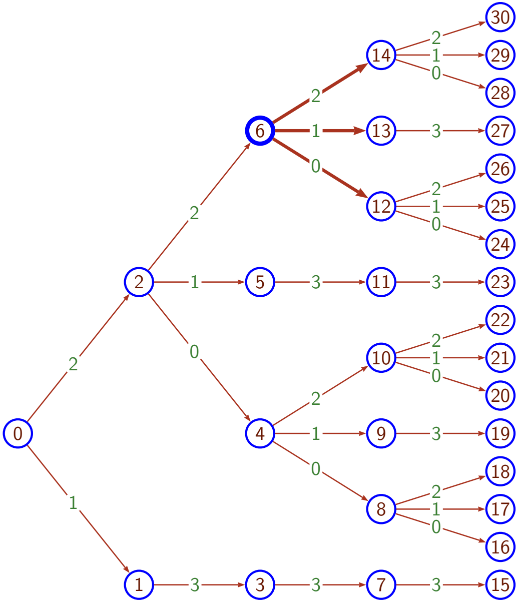

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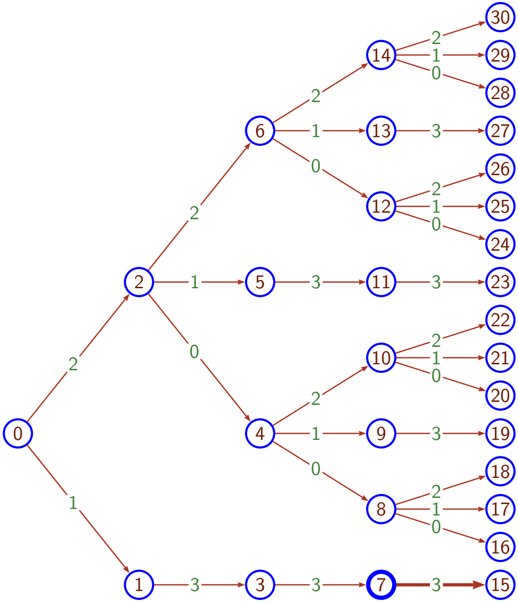

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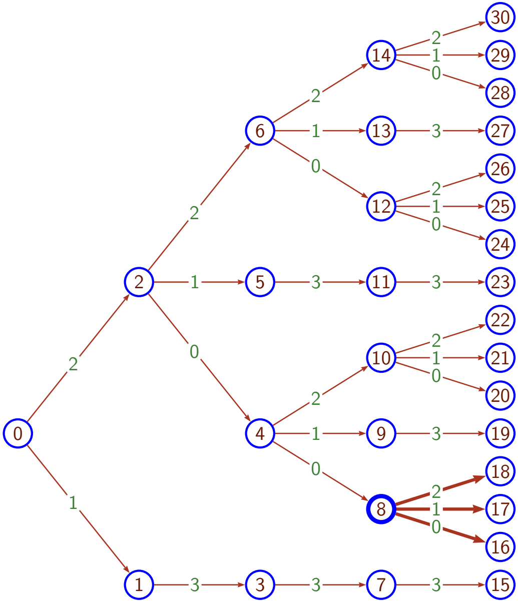

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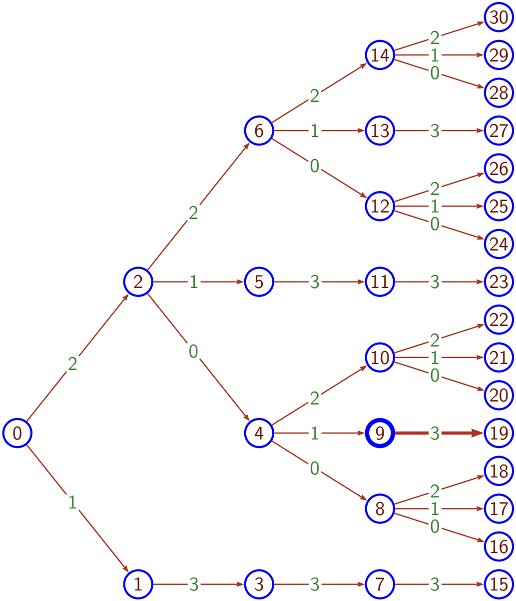

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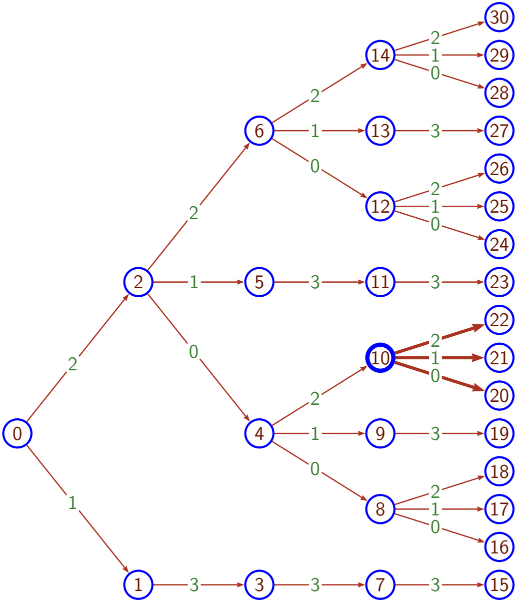

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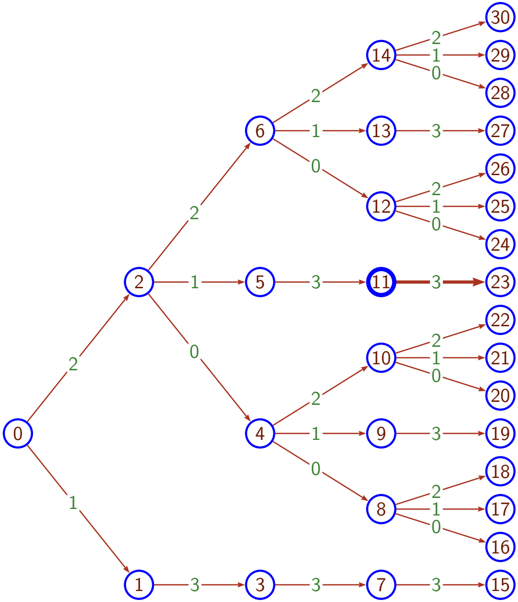

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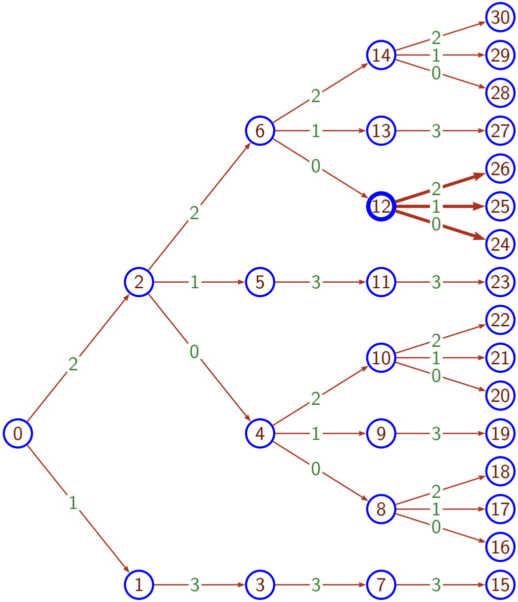

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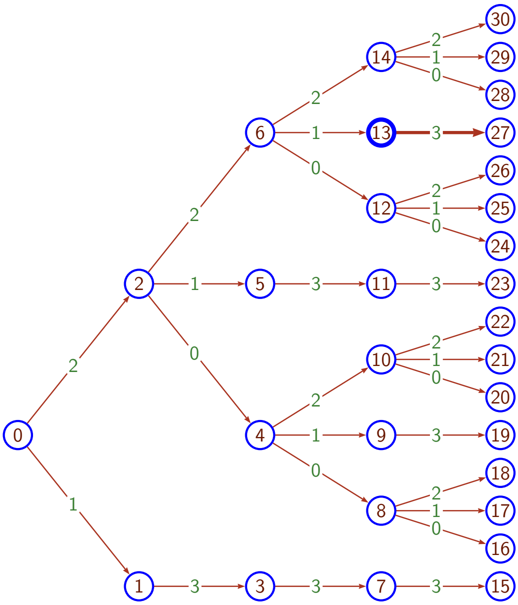

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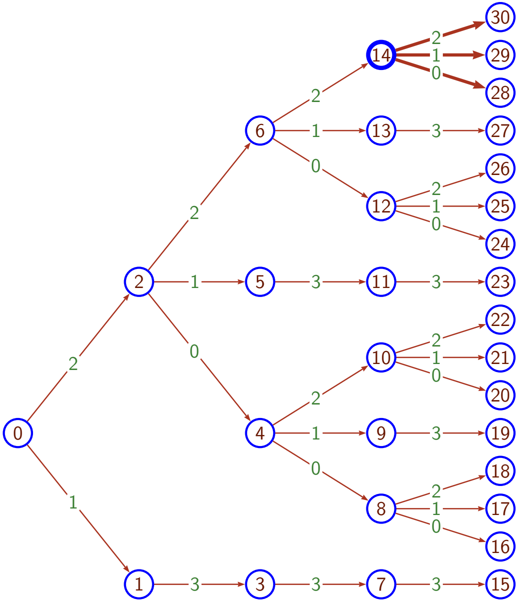

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Theorem

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- *If* $\frac{p}{q}$ *is an integer* $K_{\mathbf{r}}$ *is a rational language.*
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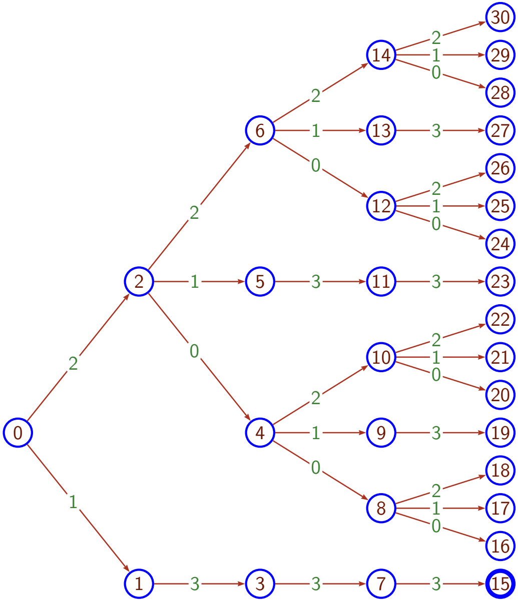
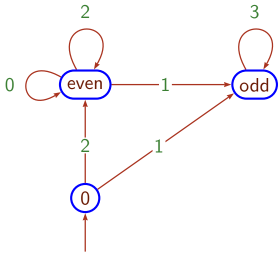
Taking back the rhythm $(3, 1)$



$$q = 2;\ p = 4;$$

$$\text{growth ratio } \frac{p}{q} = 2$$



BLIP languages

A language L is BLIP if

$$\forall u, v \quad \exists \text{ finitely many } i \quad uv^i \text{ is prefix of a word of } L$$

Example : all the prefixes of an infinite aperiodic word.

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Intuition 1

- L does not contain an infinite rational language.
  [IRS : Greibach 1975]
- L is “hard” to extend to an infinite rational language.

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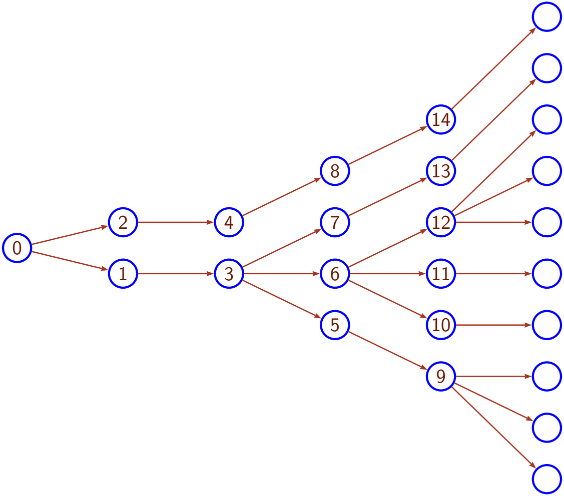
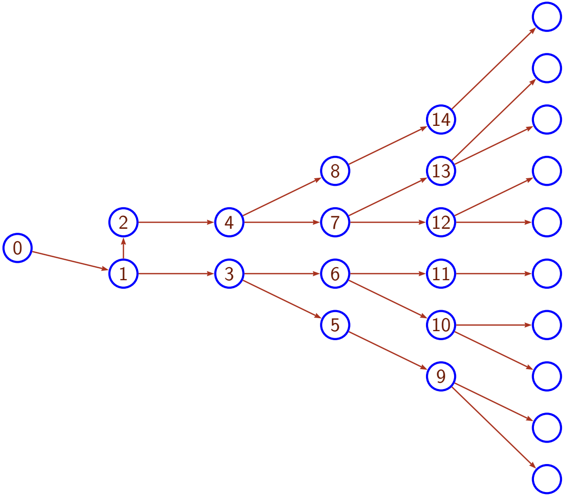
Intuition 2

- Every infinite word of the topological closure of L is aperiodic

The tree generated by $(3, 1, 1)$



The tree generated by $(2, 2, 1)$



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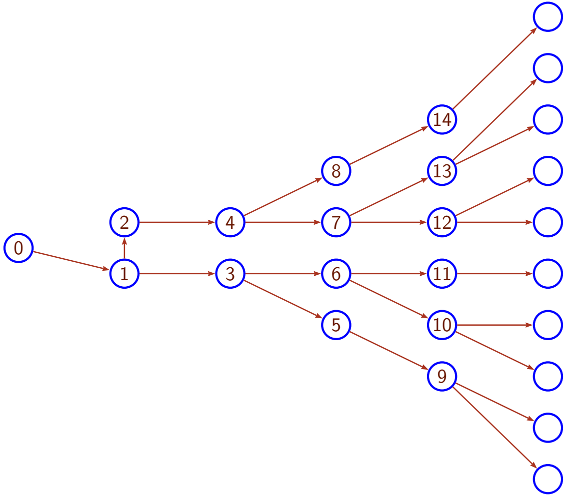


Figure: Underlying tree of the language of integers in base $\frac{5}{3}$

Quick overview of rational base number system

$\frac{p}{q}$: an irreducible fraction, or *base*

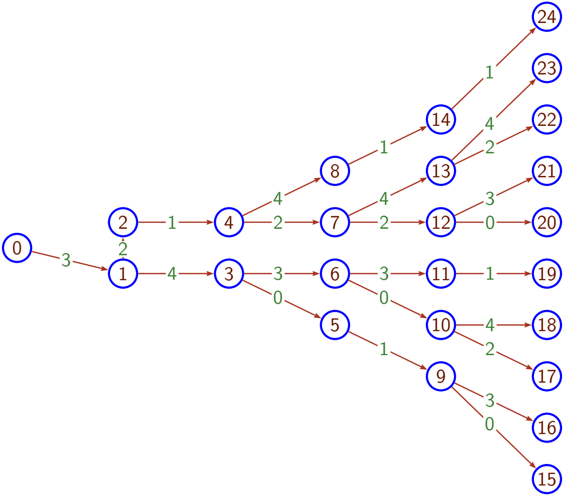
A_p: the alphabet $\{0, 1, \ldots, p-1\}$

- Evaluation: $\pi(a_n \cdots a_1 a_0) = \sum_{i=0}^{n} (\frac{a_i}{q})(\frac{p}{q})^i$.
- Each integer has a finite representation in base $\frac{p}{q}$.

$L_{\frac{p}{q}}$: the language of the representations of integers

- $L_{\frac{p}{q}}$ is prefix-closed and right-extendable.
- $L_{\frac{p}{q}}$ is a BLIP language.

The language $L_{\frac{5}{3}}$



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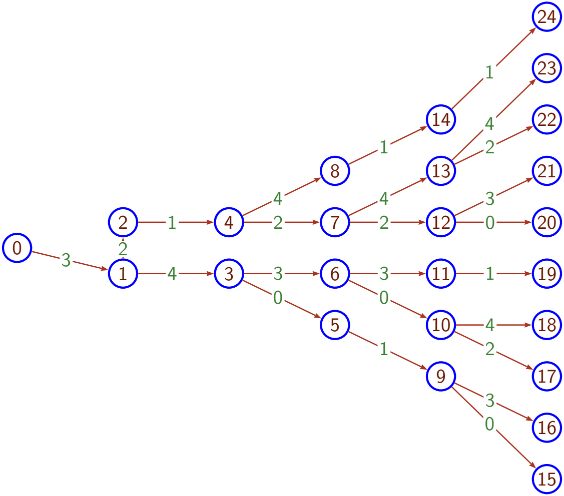

Theorem

The language $L_{\frac{p}{q}}$ is generated by Christoffel rhythm and canonical labelling.

Christoffel rhythm & Canonical labelling

Definition (Christoffel rhythm γ)

- directing parameter (q, p)
- the most equitable way to part q objects into p cases.

Example: $(2, 2, 1)$ for $\frac{5}{3}$; $(2, 1, 2, 1, 1)$ for $\frac{7}{5}$

Definition (Canonical labelling $\boldsymbol{\lambda}$)

$\boldsymbol{\lambda}$ is the p-tuple $(0, q, (2q), \ldots, (p-1)q) \quad [mod \ \ p]$

Example: $(0, 3, 2, 4, 1)$ for $\frac{5}{3}$

Conclusion

- A languages built by rhythm is a non-canonical representation of integers in either integer base or rational base;
- They are either rational or BLIP

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Perspectives

- What happens if we use infinite rhythm instead ?
  - ultimate periodic $\Longrightarrow$ probably the same as finite rhythm
  - aperiodic $\Longrightarrow$??
- What happens if we use an automata-like structure ?
- Which rational languages can be generated by rhythm ?

Example of infinite rhythms

2

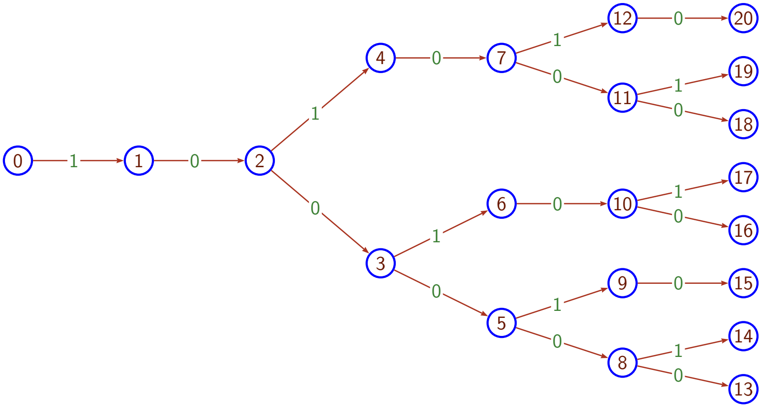
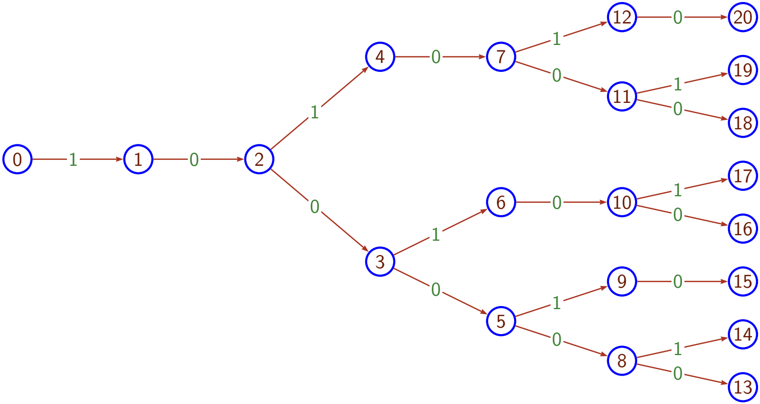


Figure: The language of the representations of integers in the Zeckendorf number system (Fibonacci)

Example of infinite rhythms

2 1



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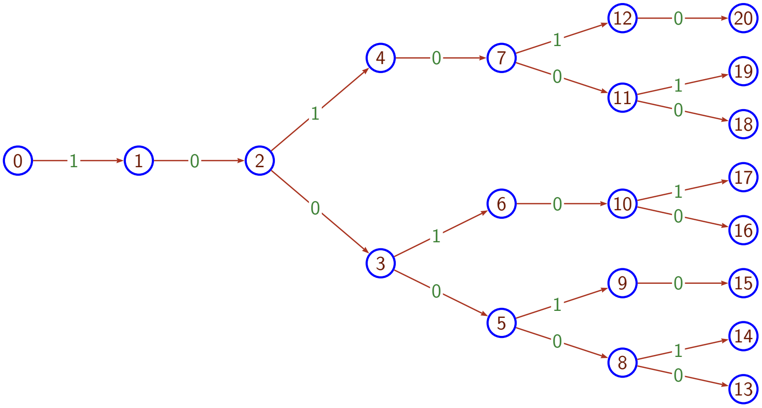
Example of infinite rhythms

2 1 2



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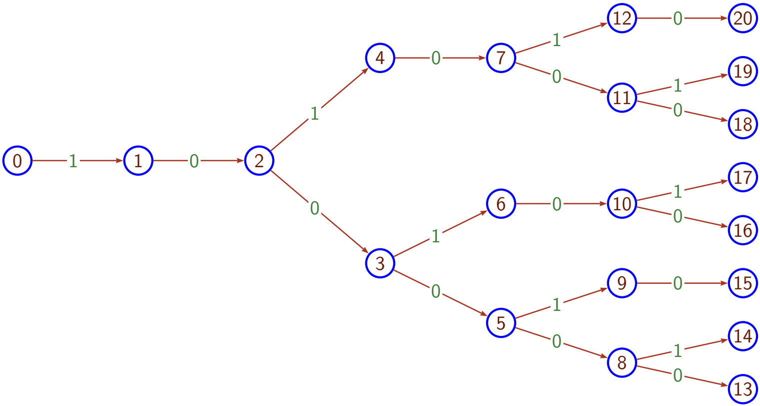

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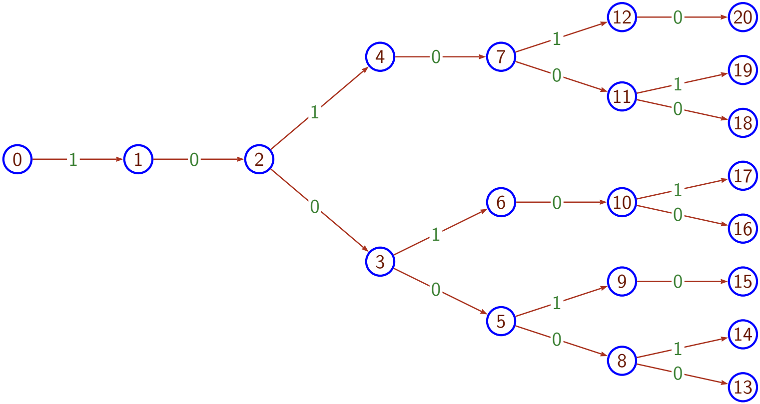
2 1 2 2 1 2 2 2 1 2 2 2 2 2 1 2 2 2 2 2 1 · · ·



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