

Auto-similarity in rational base number systems

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- 1 From integer base to rational base
- 2 The world of minimal words
- 3 Auto-similarity and derived transducer
- 4 Span of a node

- base $p \geq 2$
- alphabet $A_p = \{0, 1, \dots, p - 1\}$

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- evaluation $\pi(a_n \cdots a_1 a_0) = \sum_{i=0}^n a_i p^i$

Example (base 3) - $\pi(12) = 5$ $\pi(122) = 17$

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- $\pi(A_p^*) = \mathbb{N}$
- representation $\langle n \rangle_p = a_i \cdots a_1 a_0$ (greedy algorithm).
 - $N_0 = n$
 - $N_k = p \times N_{(k+1)} + a_k \quad \forall k > 0$
- $\langle \mathbb{N} \rangle_p = (A_p \setminus \{0\}) \cdot A_p^*$

- a base $\frac{p}{q}$ with $p > q$ and p coprime with q

- representation $\langle n \rangle_{\frac{p}{q}} = a_i \cdots a_1 a_0 :$

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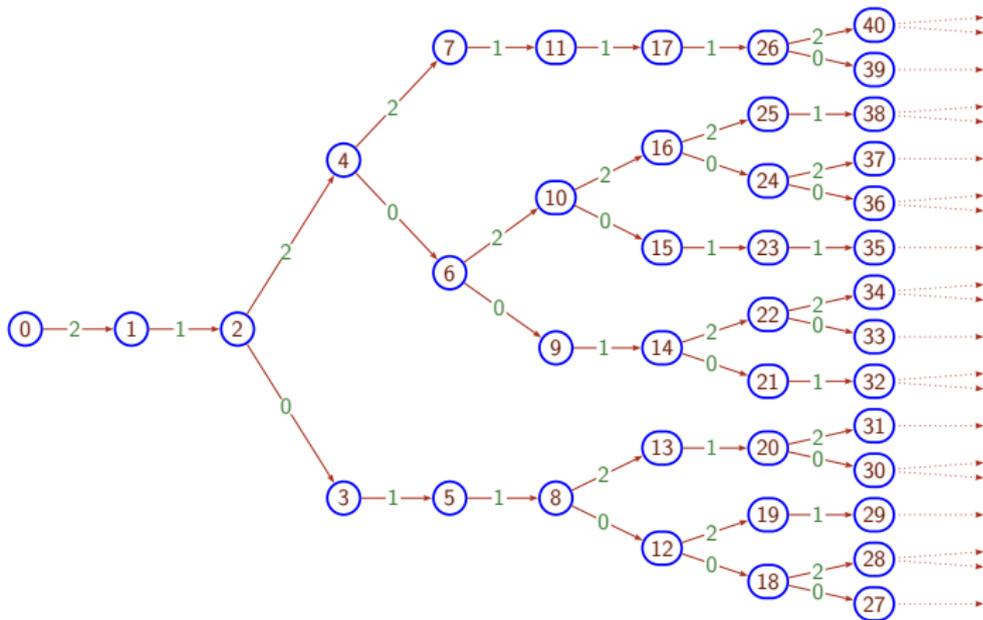
Example ($\frac{3}{2}$ -system) $\pi(2) = 1$ $\pi(20) = \frac{3}{2}$ $\pi(21) = 2$

- representation $\langle n \rangle_{\frac{p}{q}} = a_i \cdots a_1 a_0 :$

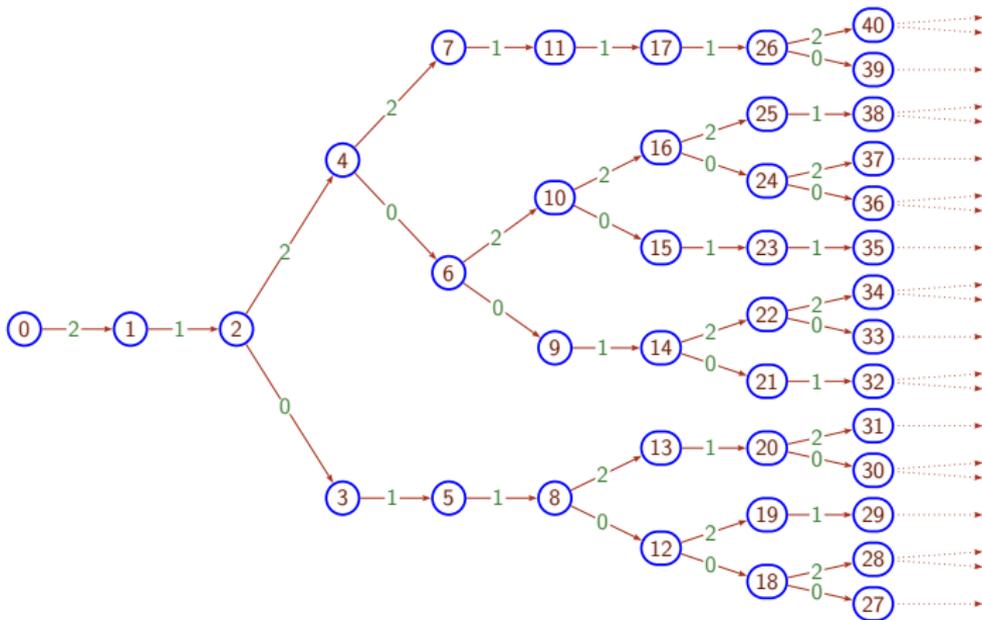
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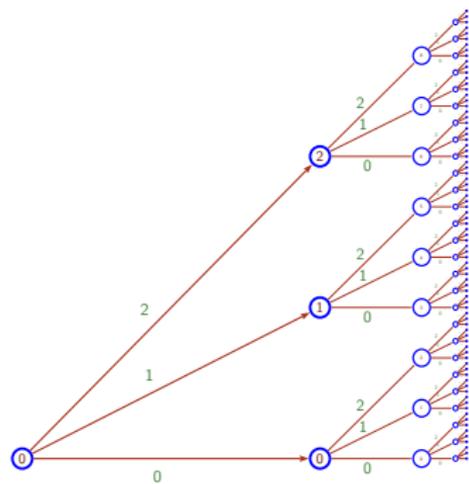
- a base $\frac{p}{q}$ with $p > q$ and p coprime with q
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- evaluation $\pi(a_n \cdots a_1 a_0) = \sum_{i=0}^n \left(\frac{a_i}{q}\right) \left(\frac{p}{q}\right)^i$
- $\mathbb{N} \subsetneq \pi(A_p^*) \subsetneq \mathbb{Q}$
- representation $\langle n \rangle_{\frac{p}{q}} = a_i \cdots a_1 a_0 :$
 - $N_0 = n$
 - $q \times N_k = p \times N_{(k+1)} + a_k \quad \forall k > 0$
- $L_{\frac{p}{q}} = \langle \mathbb{N} \rangle_{\frac{p}{q}}$

- $L_{\frac{p}{q}}$ is prefix-closed and right-extendable.

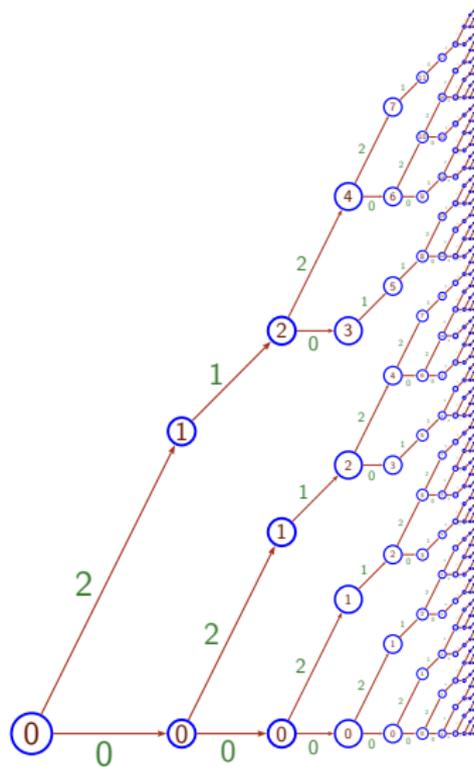


- $L_{\frac{p}{q}}$ is prefix-closed and right-extendable.
- $L_{\frac{p}{q}}$ is not rational (not even context-free) [AFS'08].

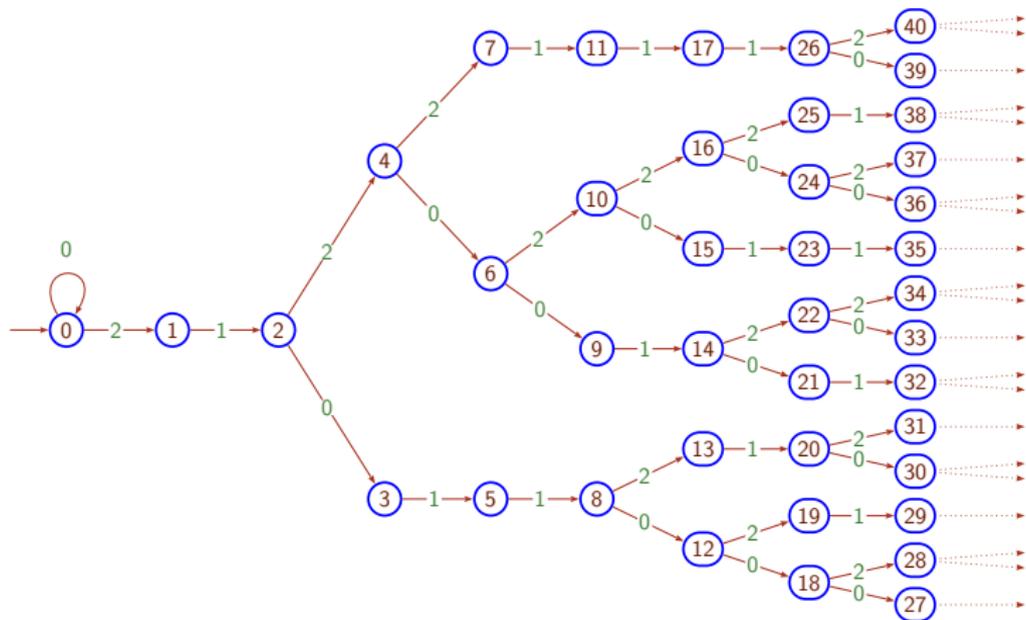




(a) Integer base 3

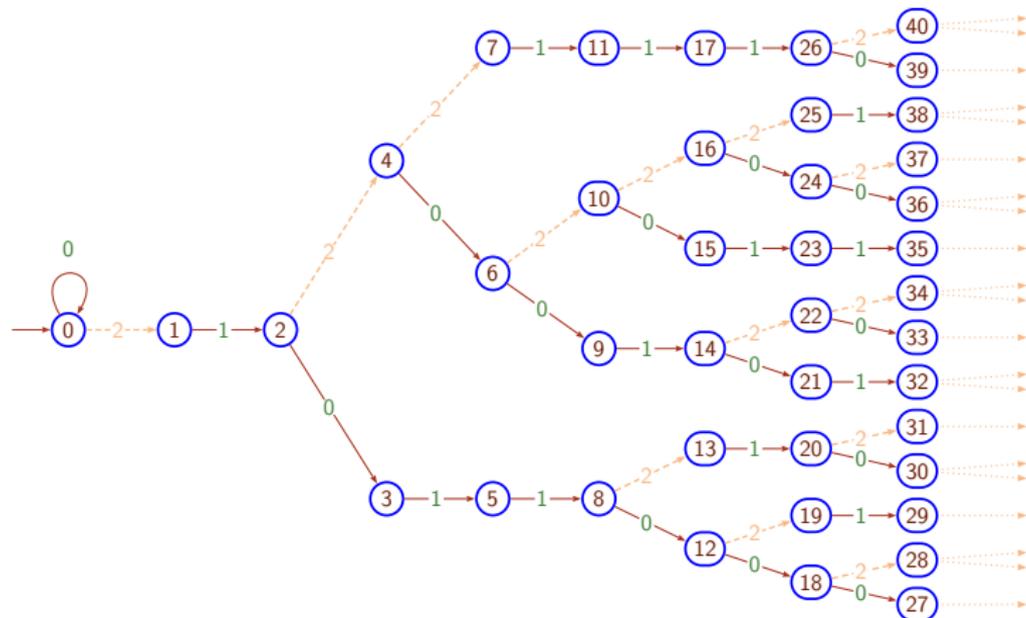


(b) Rational base $\frac{3}{2}$



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w_n : the (infinite) word starting from n taking the lowest branch.



Given an integer n , the minimal word w_n is

- over the alphabet $\{0, \dots, (q - 1)\} = A_q$
- the unique word over A_q readable from n

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Proposition (Akiyama-Frougny-Sakarovitch, 2008)

w_n and w_m have the same prefix of length k .

$$\begin{array}{c} \updownarrow \\ n \equiv m [q^k] \end{array}$$

Given an integer n , the minimal word w_n is

- over the alphabet $\{0, \dots, (q - 1)\} = A_q$
- the unique word over A_q readable from n
- different from w_m (for $m \neq n$)
- aperiodic

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$$\begin{array}{c} \updownarrow \\ n \equiv m [q^k] \end{array}$$

W^- : the set of minimal words.

Topological properties

- The topological closure of W^- is A_q^ω whole.
- The interior of W^- is *empty*.

Shift operation

- W^- is stable by shift
- W^- cannot be finitely generated through shift.

$$\begin{array}{l} \gamma : A_q^\omega \longrightarrow A_q^\omega \\ w_n \longmapsto w_{n+1} \end{array}$$

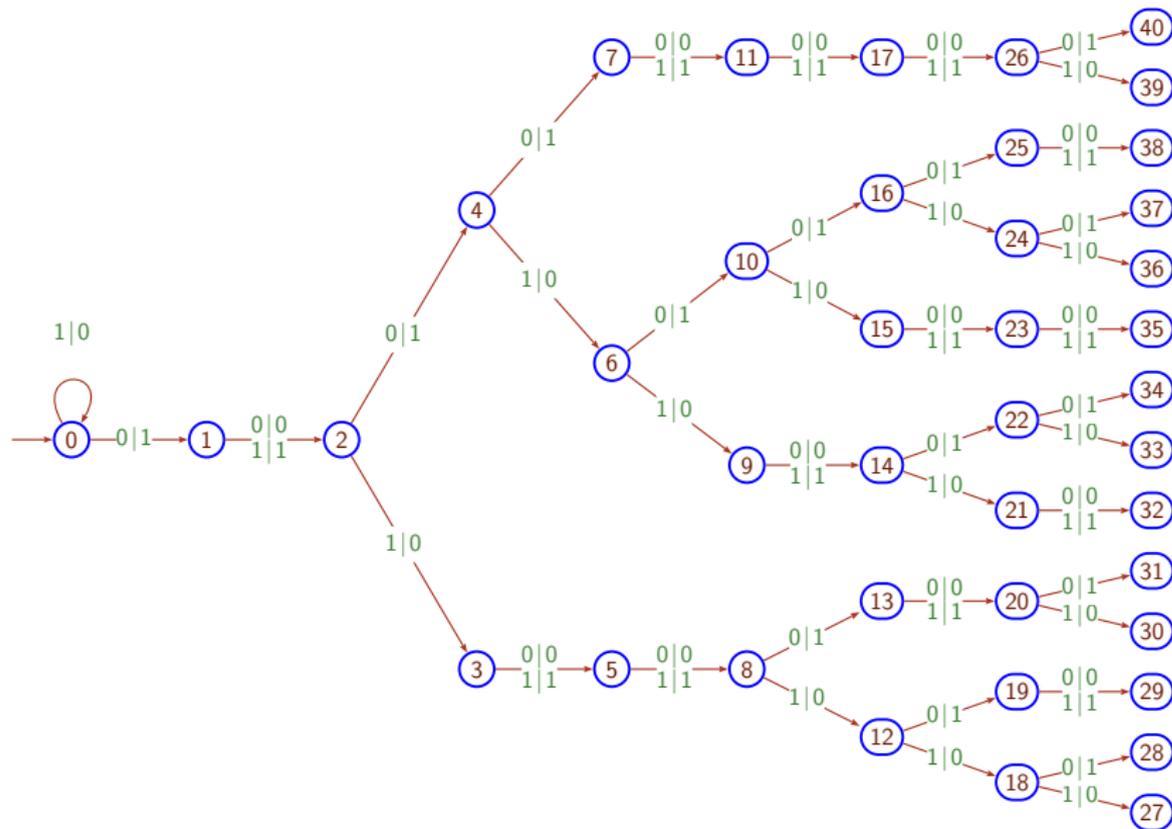
$$\begin{aligned} \gamma : A_q^\omega &\longrightarrow A_q^\omega \\ w_n &\longmapsto w_{n+1} \end{aligned}$$

Remark

$$\forall n \exists i \quad \gamma^i(w_n) = \sigma(w_n).$$

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Derived transducer for $\frac{3}{2}$: $D_{\frac{3}{2}}$



Proposition

If $p = 2q - 1$,

- the underlying graph of $D_{\frac{p}{q}}$ and $T_{\frac{p}{q}}$ are identical;
- the labels of the transitions of $D_{\frac{p}{q}}$ are obtained by an (injective) substitution from those of $T_{\frac{p}{q}}$.

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Theorem

The derived transducer $D_{\frac{p}{q}}$ is locally computable from $T_{\frac{p}{q}}$.

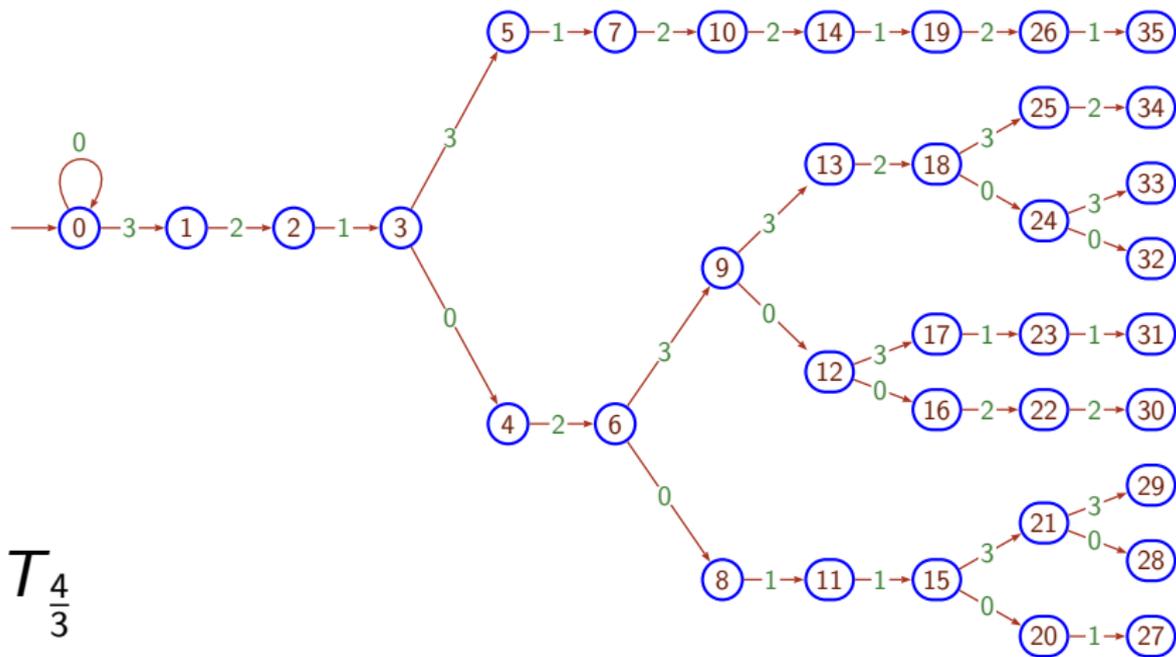
Step 1: changing the alphabet

$$A_p \longrightarrow B_{p,q} = \{p - (2q - 1), \dots, p - 1\}$$

- if $p = (2q - 1)$, $A_p = B_{p,q}$
- if $p < (2q - 1)$, $A_p \subseteq B_{p,q}$ (the base $\frac{p}{q}$ is “too small”)
- if $p > (2q - 1)$, $A_p \supseteq B_{p,q}$ (the base $\frac{p}{q}$ is “too big”)

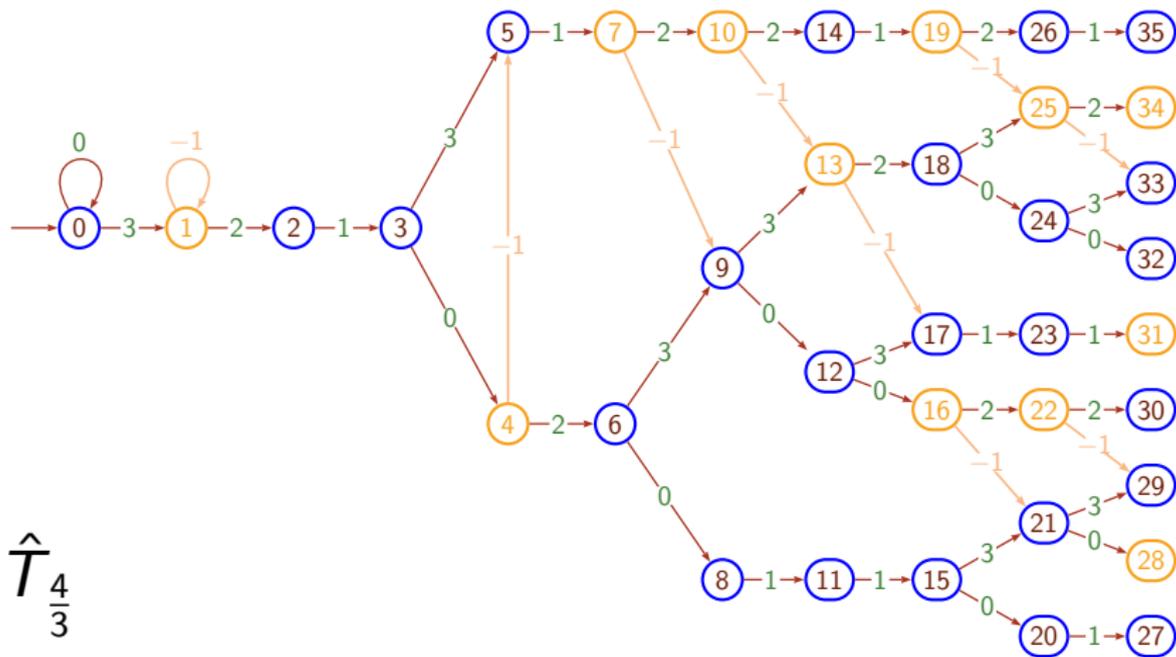
Example of the “small” base $\frac{4}{3}$ (Step 1)

■ $A_4 = \{0, 1, 2, 3\} \subseteq \{-1, 0, 1, 2, 3\} = B_{4,3}$



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$$\hat{T}_{\frac{4}{3}}$$

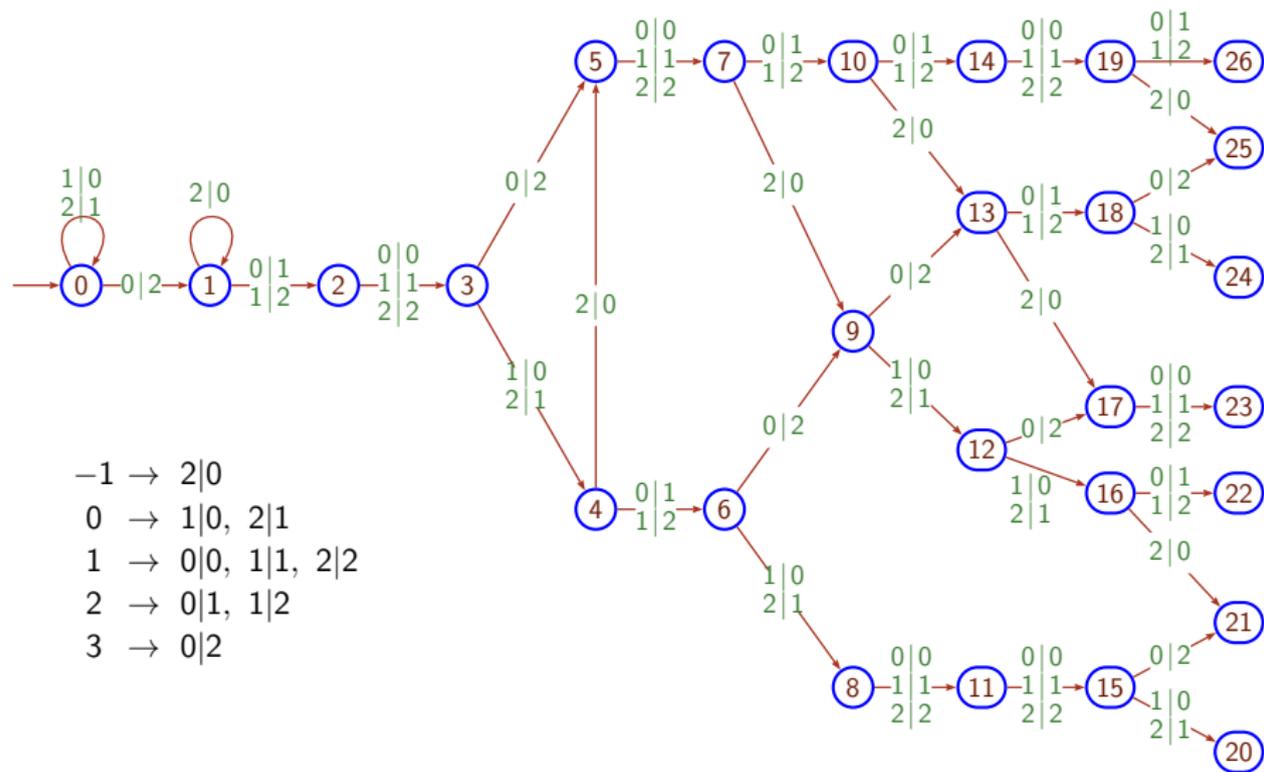
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Step 2: changing the labels

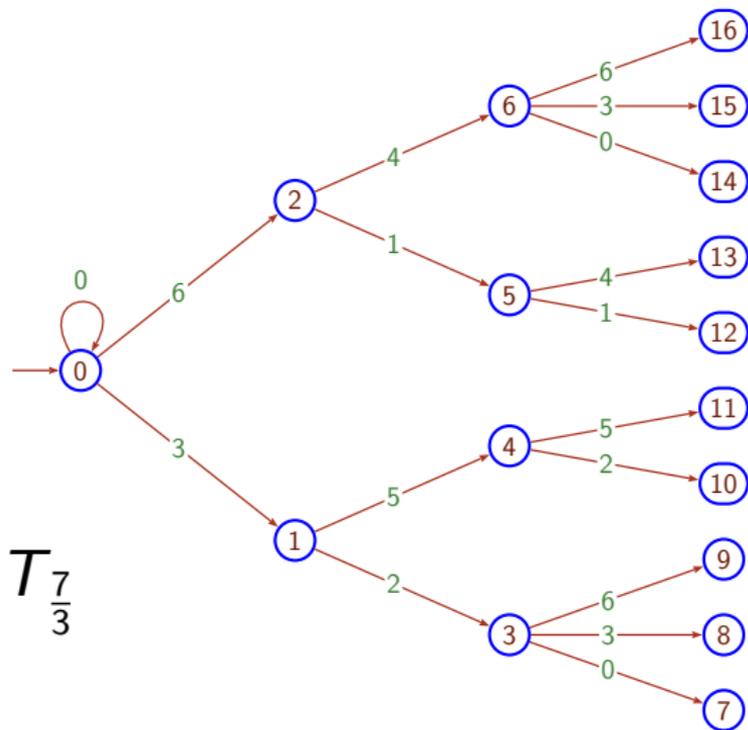
$$\begin{aligned} \omega : B_{p,q} &\longrightarrow \mathbb{P}(A_p \times A_p) \\ a &\longmapsto \{(b|c) \mid (b - c) = \underbrace{a - (p - q)}_{\text{distance to the center of B}}\} \end{aligned}$$

Example of the “small” base $\frac{4}{3}$ (Step 2)



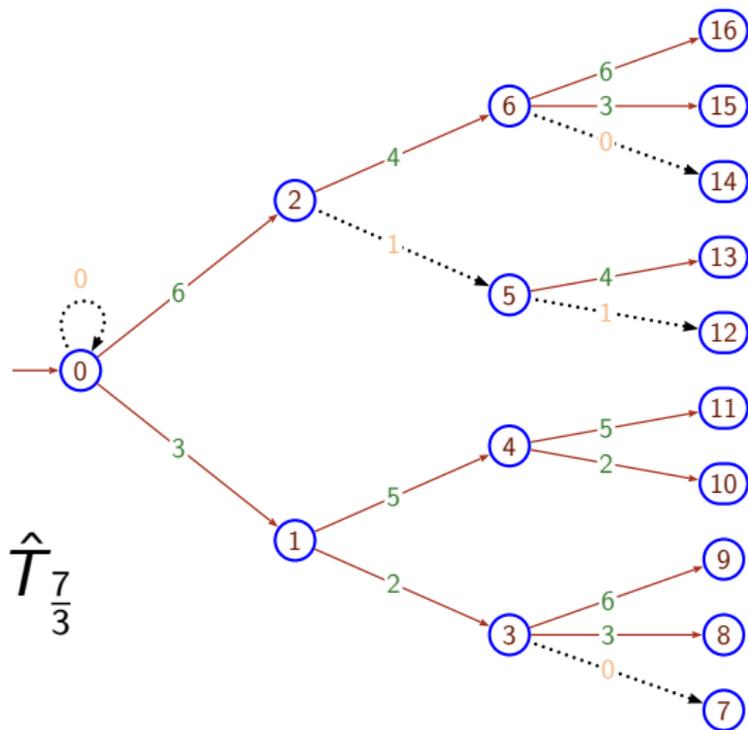
Example of the “big” base $\frac{7}{3}$ (Step 1)

- $A_7 = \{0, 1, 2, 3, 4, 5, 6\} \supseteq \{2, 3, 4, 5, 6\} = B_{7,3}$



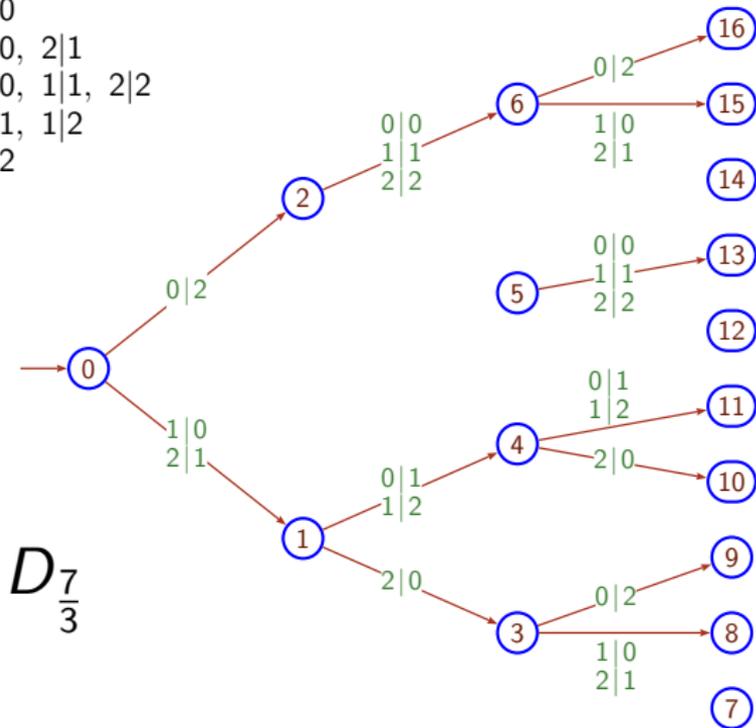
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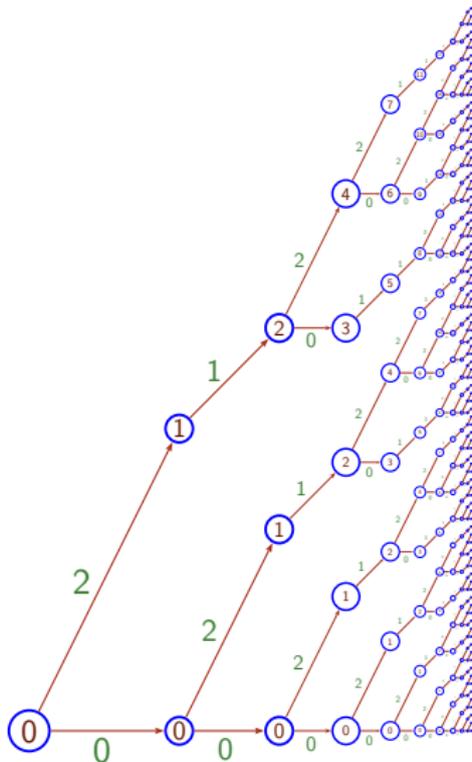
Example of the “big” base $\frac{7}{3}$ (Step 2)

- 2 \rightarrow 2|0
- 3 \rightarrow 1|0, 2|1
- 4 \rightarrow 0|0, 1|1, 2|2
- 5 \rightarrow 0|1, 1|2
- 6 \rightarrow 0|2



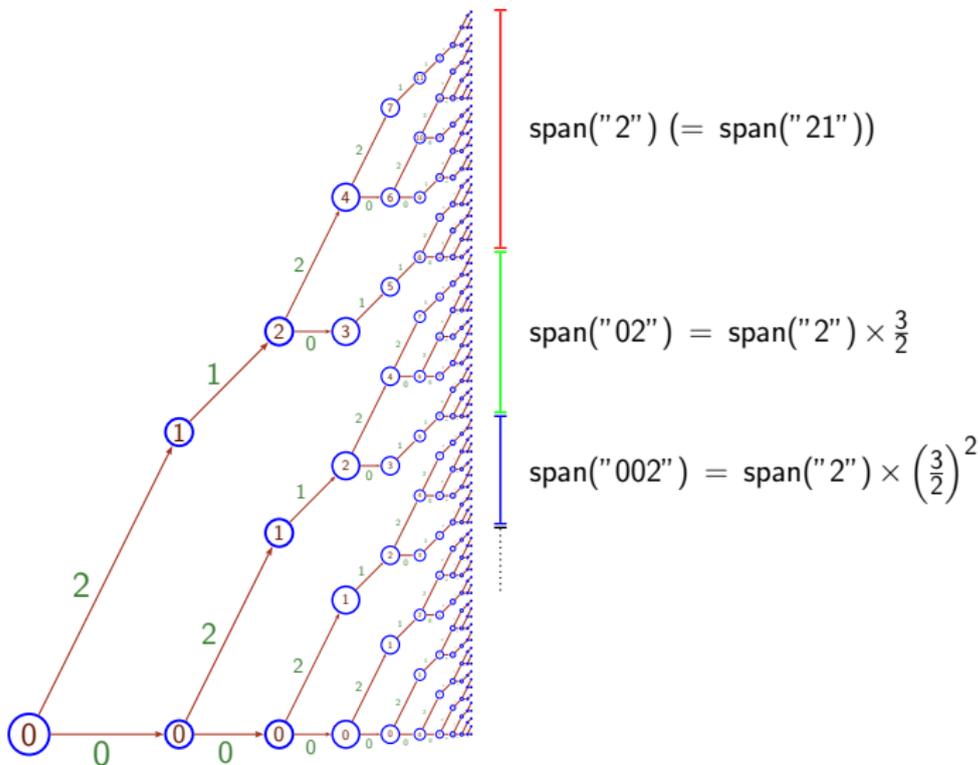
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$$\rho(a_1 a_2 \cdots a_n \cdots) = \sum_{i \geq 0} \frac{a_i}{q} \left(\frac{p}{q}\right)^{-i}.$$



Definition – span of the node X

The length of the interval reachable from X in the tree.



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Definition – renormalised span of the node X

the span of X multiplied by $(\frac{p}{q})^k$, where k is the depth of X .

$S_{\frac{p}{q}}$ denotes the set of the renormalised span of every node.

Definition

- The span of n is represented by the word $(w'_n \ominus w_n)$, where:
 - w'_n is the *maximal* word starting from n ;
 - “ \ominus ” denotes the digit-wise subtraction.
(Example : $321 \ominus 012 = 31(-1)$)
- It is called the *span-word* of n and is over the alphabet $B_{p,q}$.

Proposition

$\hat{T}_{\frac{p}{q}}$ accepts the topological closure of the language of the span-words.

Theorem

- If $p \leq 2q - 1$, $S_{\frac{p}{q}}$ is dense.
- If $p > 2q - 1$, $S_{\frac{p}{q}}$ is nowhere dense.

- The derived transducer somehow requires the same structure as the original tree.
- The topological properties of the set of spans divides the rational base number systems in two classes.
- The cases $p = 2q - 1$ is remarkable in both constructions.

Next question

For a given integer n ,
is there a *finite* transducer realising $w_n \mapsto w_{n+1}$?