Signature and Numeration Systems

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Based on several joint work with Jacques Sakarovitch:

M15 M16 MS17a MS17b

Outline

Numeration Systems

Numbers

- Abstract quantities
- Axiomatised in Math

Ex: $8,\ \sqrt{2},\ \pi$, *etc*.

Representation

Words

- Sequence of letters/digits
- Used for effective computation

Ex: $abbac,\ 3.141592\cdots,$

Evaluation

The Three Components of a NS

Alphabet

Authorised digits

Evaluation function

- word $\mapsto$ number
- the value of a word u is written $\pi(u)$

Representation function

- number $\mapsto$ word
- the representation of a number n is written $\langle n \rangle$

Ex: Our Usual System (Base 10)

Alphabet

$$\{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}$$

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 $\mapsto$ 19 (a digit 1 followed by a digit 9)

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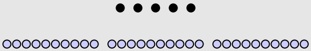

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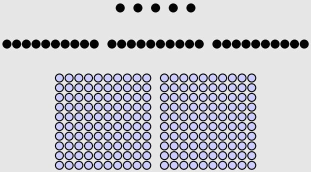

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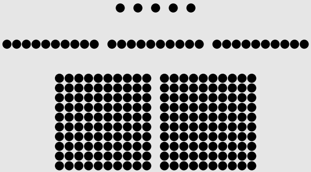
2 3 5 $\rightarrow$ 

Two Ways to Define a NS

Define the evaluation (Concrete NS)

- Choose how to evaluate a word: $a_n \cdots a_1 \mapsto f(a_n, \cdots, a_1)$
  where f is an arithmetic function
- Among words with the same value, choose a canonical one

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Define the representation (Abstract NS, Lecomte-Rigo '01)

- Choose a language L of representations
- Choose an order for the alphabet of L
- The n-th word of L is the representation of n:
  - a shorter word is smaller than a longer word;
  - two word of the same length are ordered lexicographically.

Almost all concrete NS are also abstract NS.

Example of Concrete NS: Fibonacci (Zeckendorf)

- Based on the sequence: $F_0 = 1,\ F_1 = 2,\ F_{n+2} = F_{n+1} + F_n$
- Alphabet: $\{0,1\}$
- Evaluation: $a_n \cdots a_0 \mapsto \sum_{k=0}^{n} a_k F_k$

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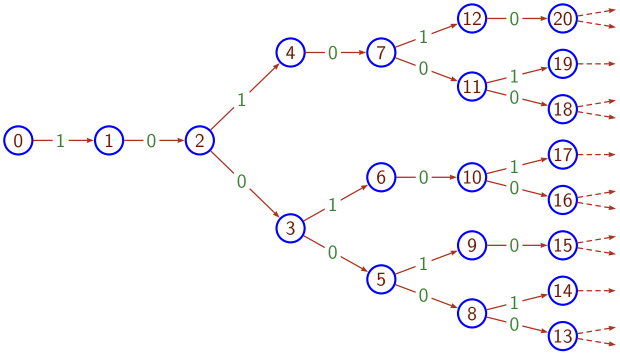
- Representation of $\mathbb{N}$: $(10+0)^*(1+\varepsilon)$



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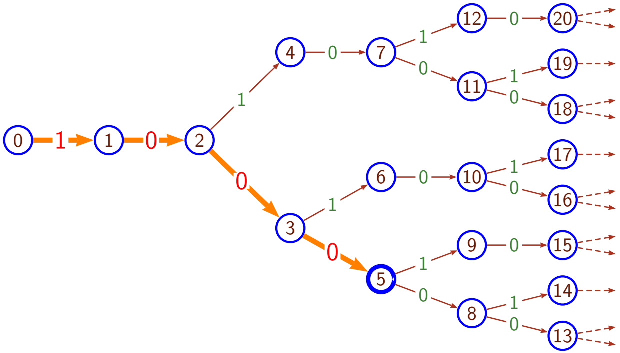

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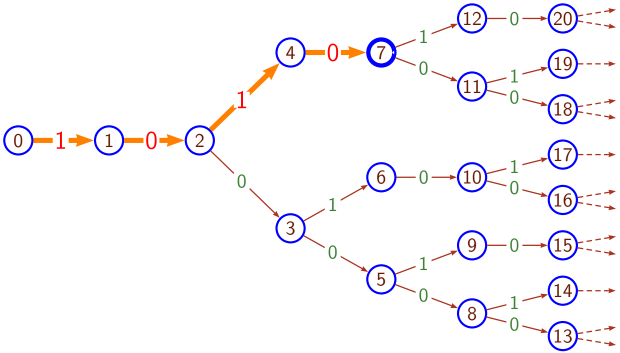

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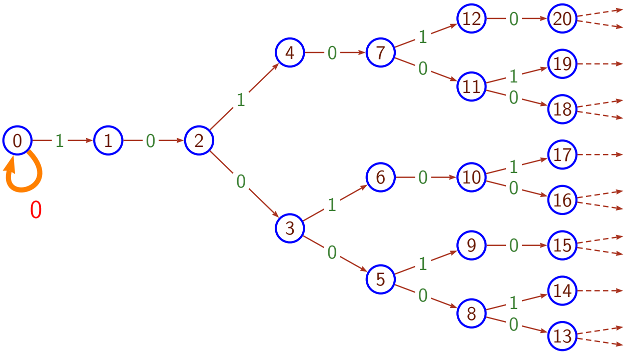
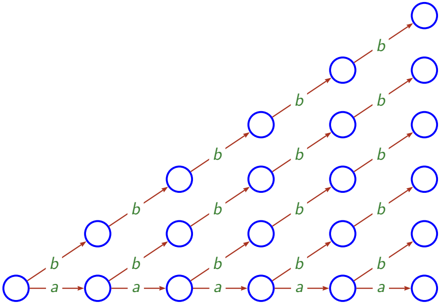
- Natural padding letter: 0



Example of Abstract NS: a^*b^*

- Alphabet: $\{a, b\}$ ordered $a < b$
- Representation of $\mathbb{N}$: a^*b^*



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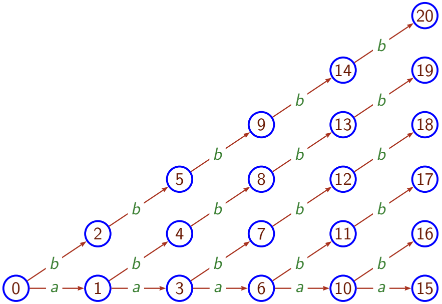

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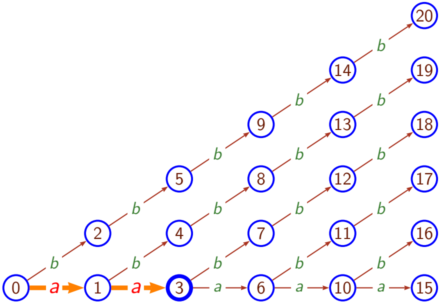

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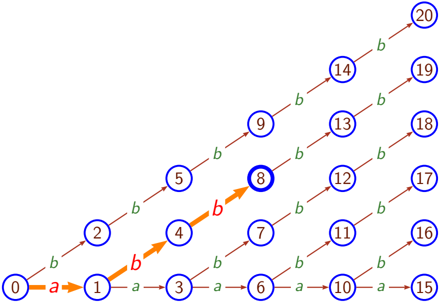

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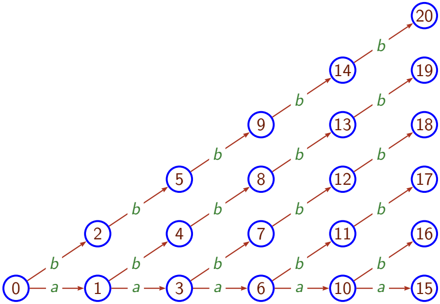
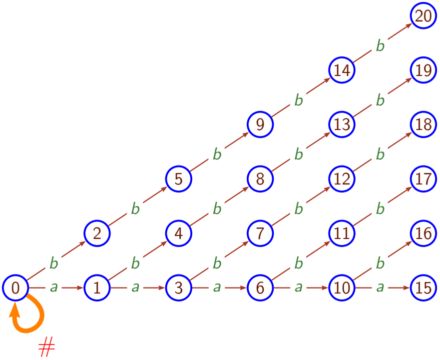


Example of Abstract NS: a^*b^*

- Alphabet: $\{a, b, \#\}$ ordered $\# < a < b$
- Representation of $\mathbb{N}$: a^*b^*
- Evaluation: $a_n \cdots a_1 \mapsto$???
- Padding letter: assume one or add one.



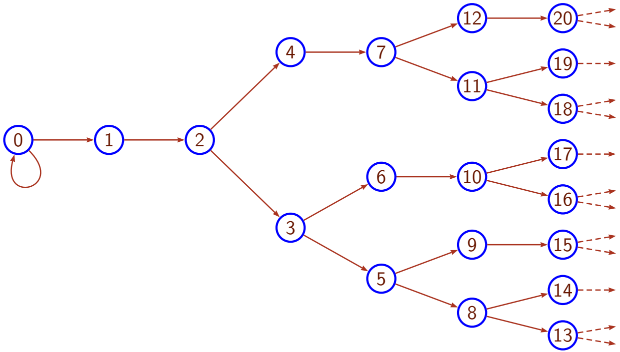
A few details

- We assume alphabets to be **ordered**
- We assume languages to be **prefix-closed**
- We assume languages to be **padded** :
  - there is a padding letter $\#$ such that $\#^*L = L$;
  - the padding letter is the least in the alphabet.
- We consider regular ANS's and nonregular ANS's

Outline

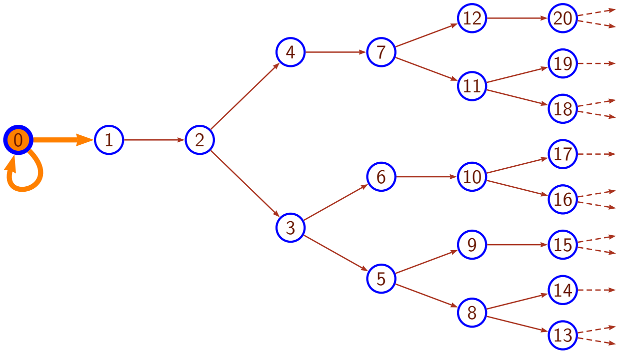
Signature of a tree

- Let's forget about letters for a moment.
- We obtain a (rooted, ordered, infinite) tree with a loop.
- Such a tree has a canonical breadth-first traversal.



Definition

The **signature** of a tree is the sequence of the degree of the nodes taken in breadth-first order.



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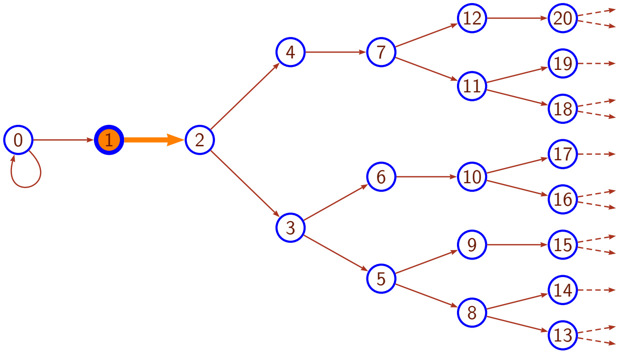

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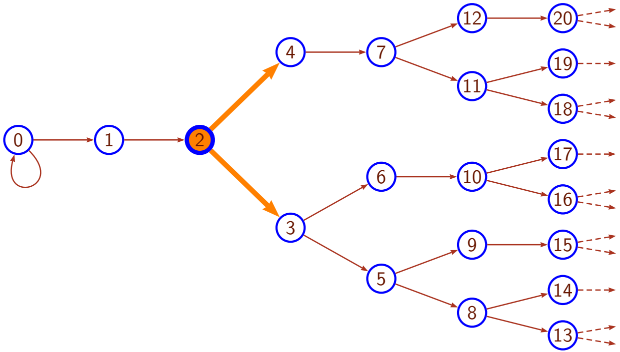

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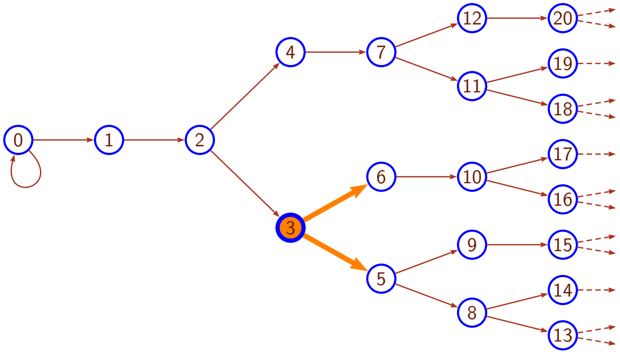

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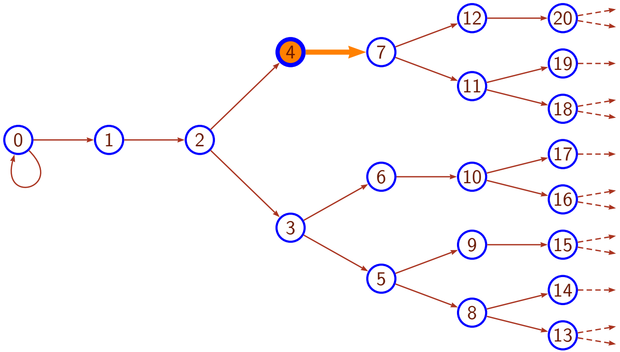

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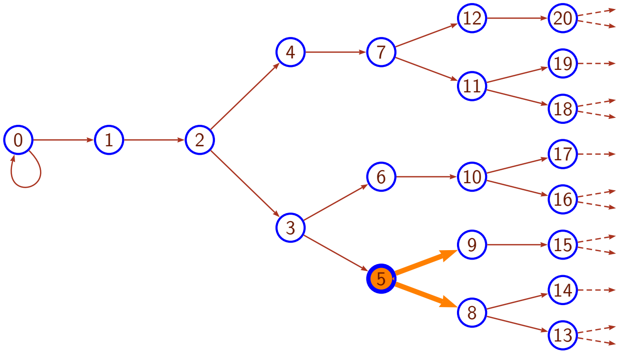

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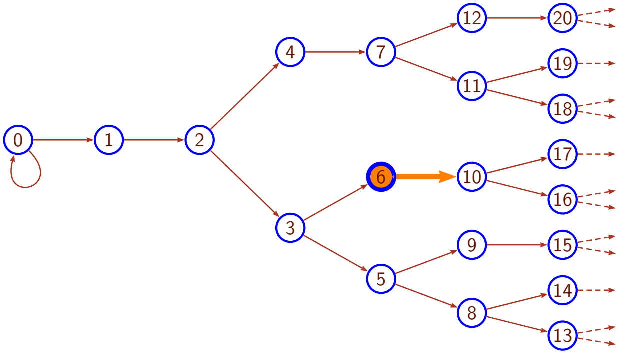

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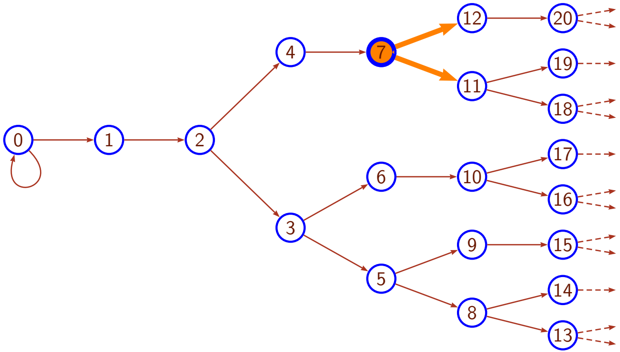

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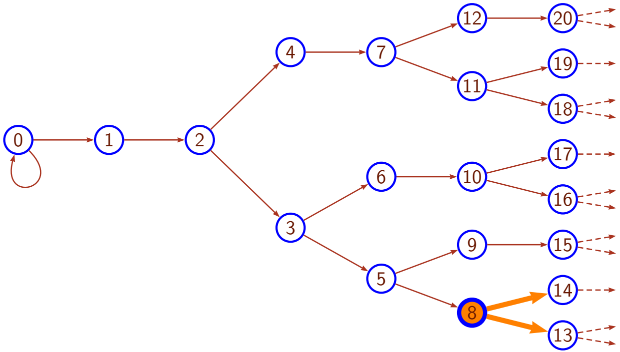

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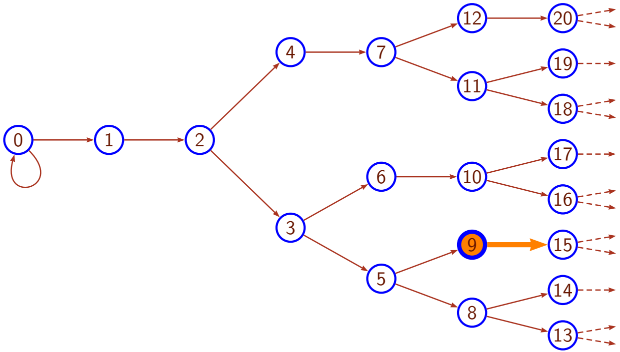

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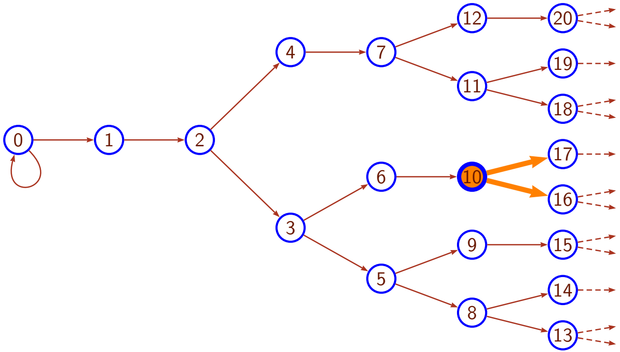

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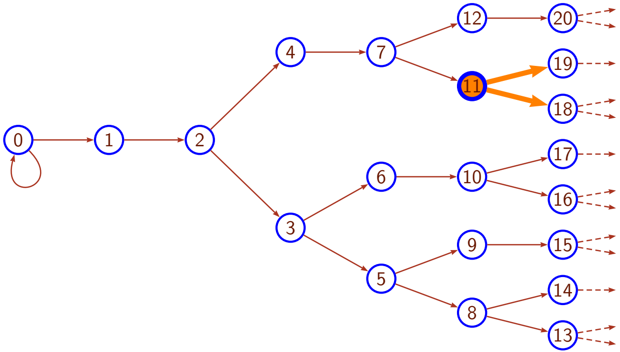

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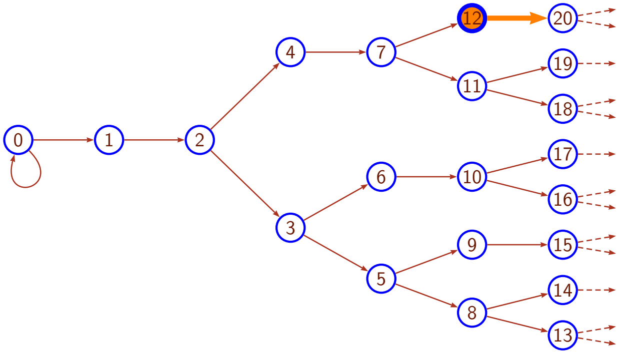

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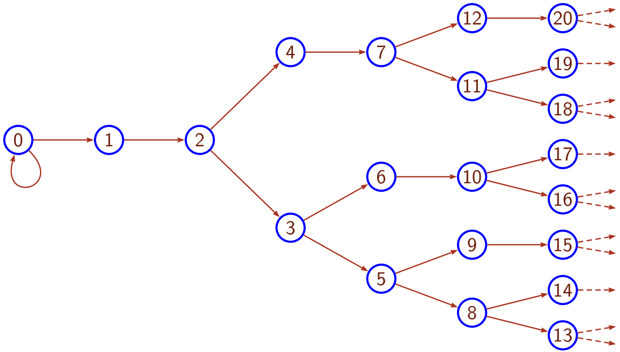

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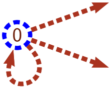
Signature is characteristic of tree

$$\mathbf{s} = (\ \boxed{3}\ \ 2\ \ 1\ )^{\omega}$$



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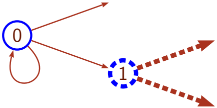

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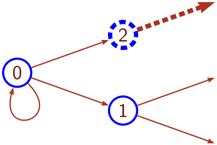

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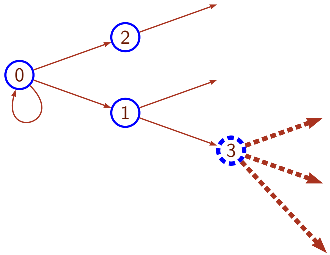

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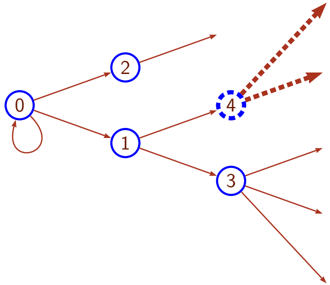

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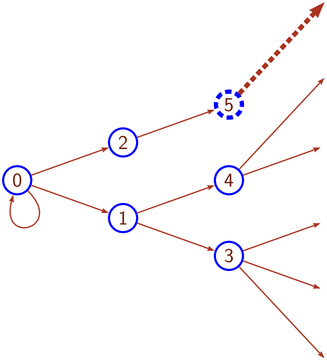

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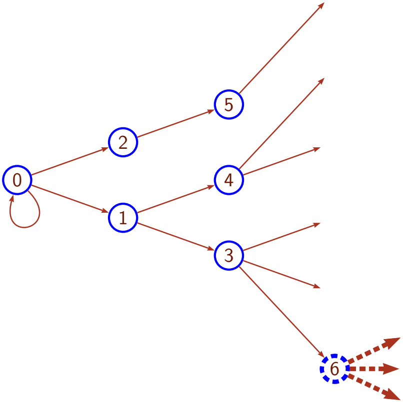

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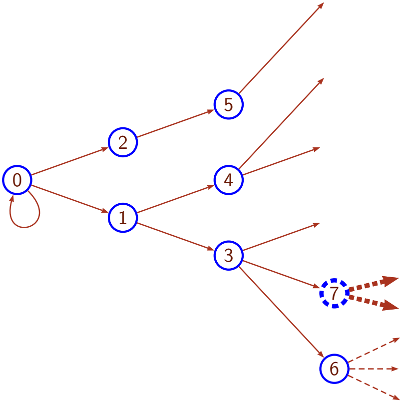

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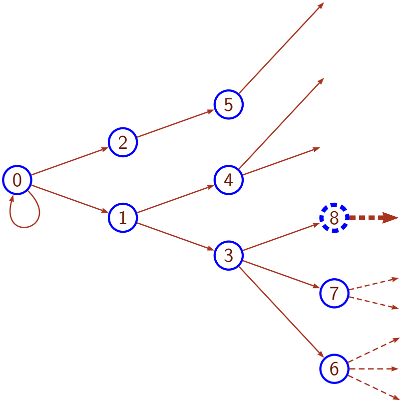

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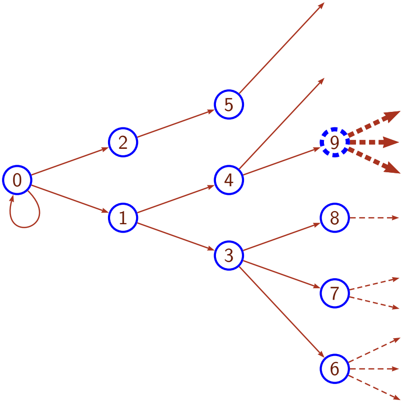

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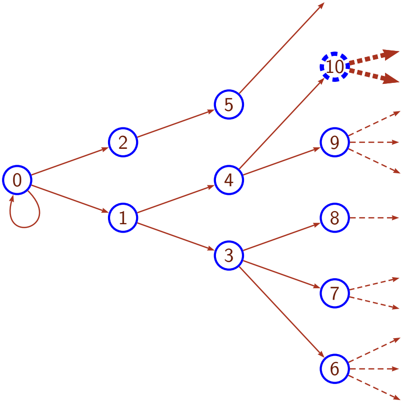

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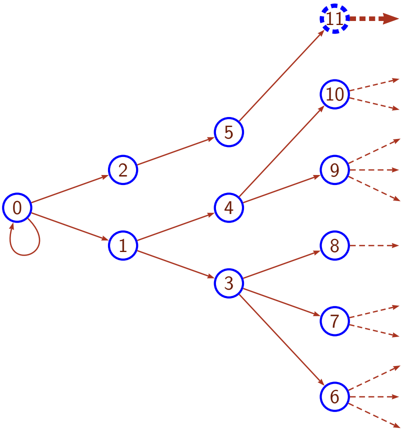

Definition

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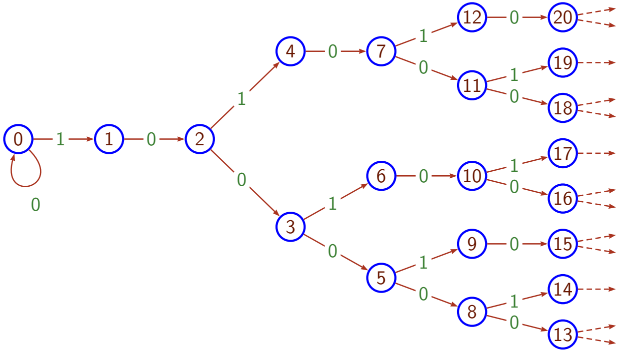

$\mathbf{s} =$

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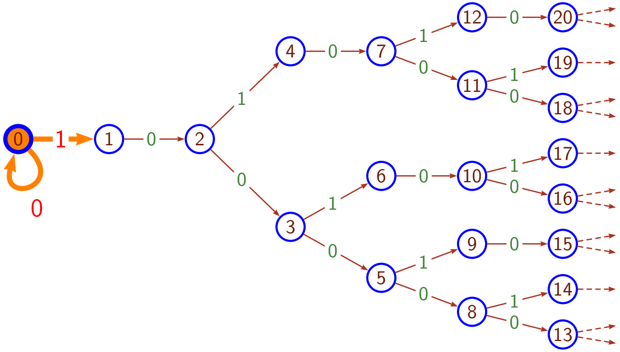

$\mathbf{s} = \boxed{2}$

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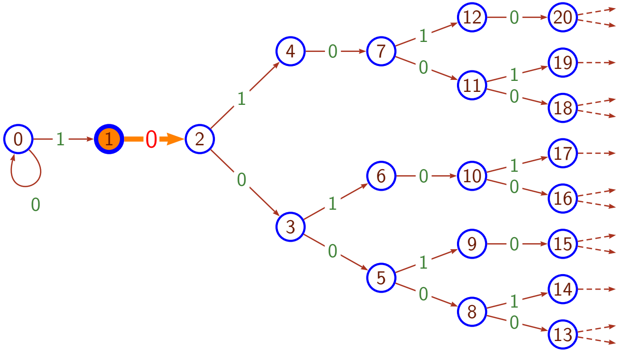

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Serialisation of a prefix-closed language

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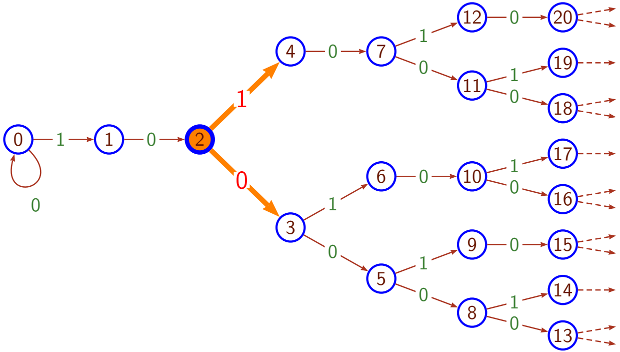

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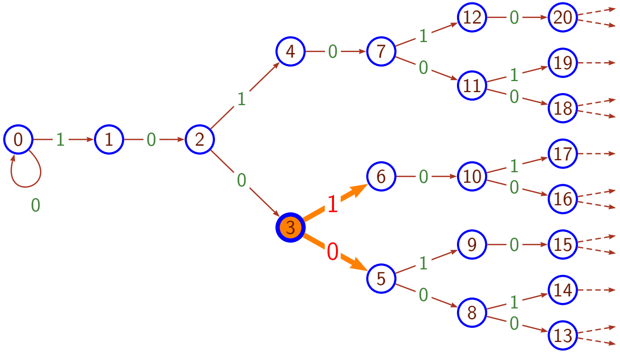

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$$\lambda = 01\ 0\ 01\ 01$$

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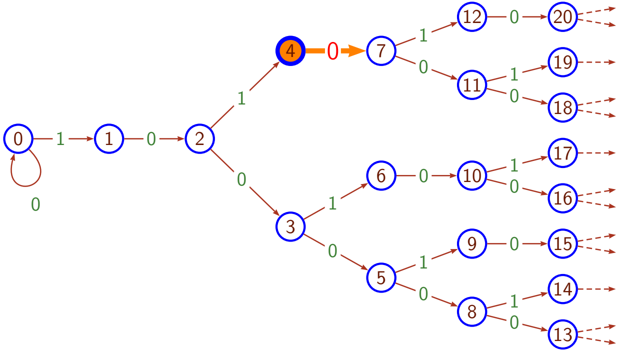

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Serialisation of a prefix-closed language

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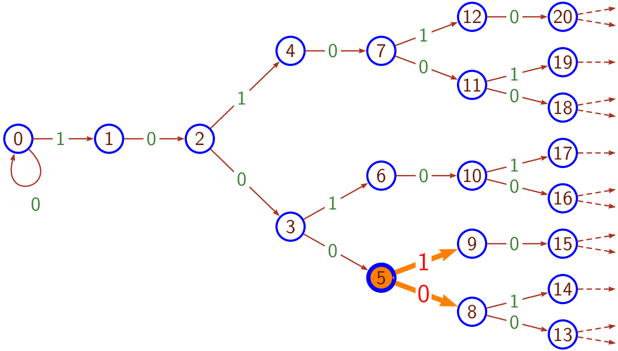

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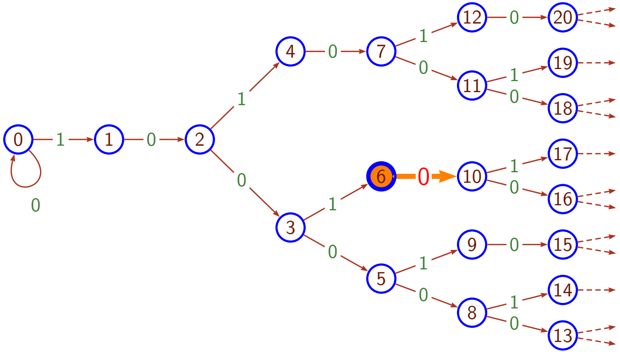

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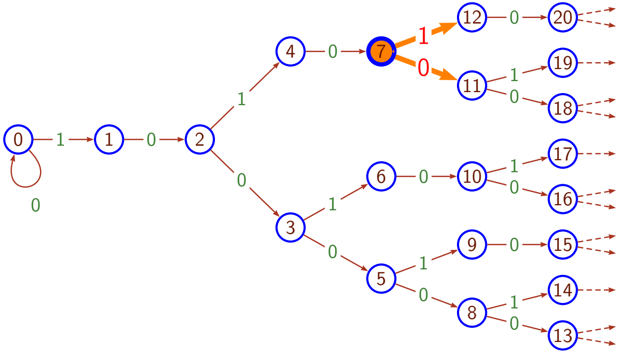

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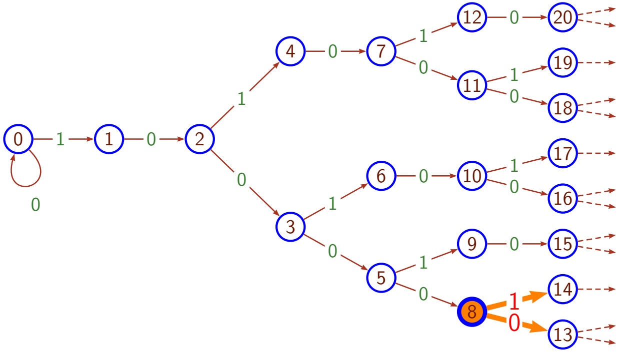

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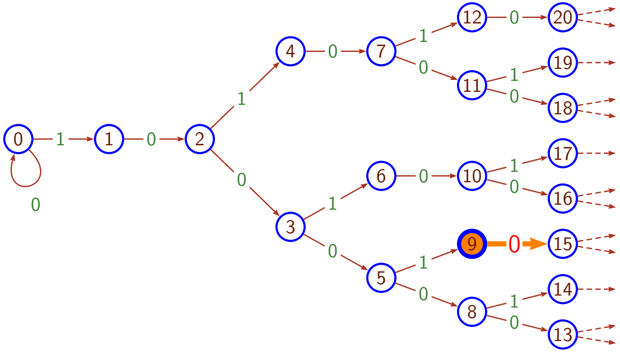

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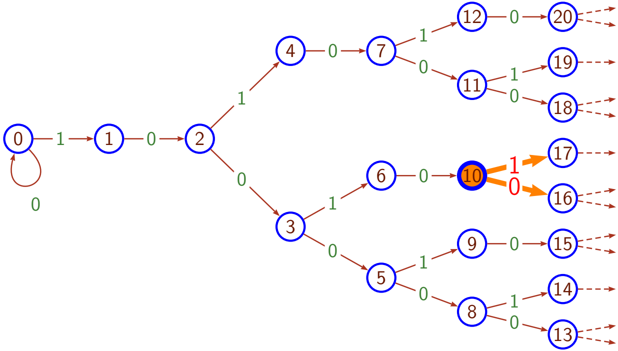

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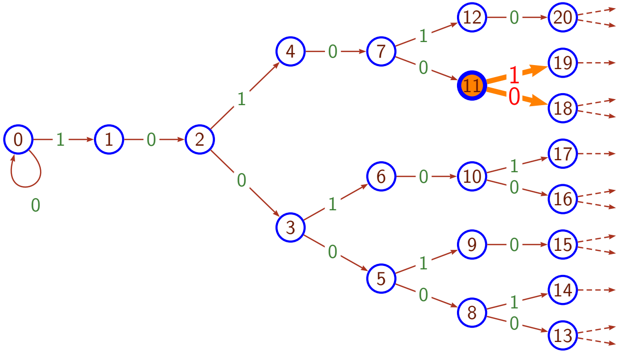

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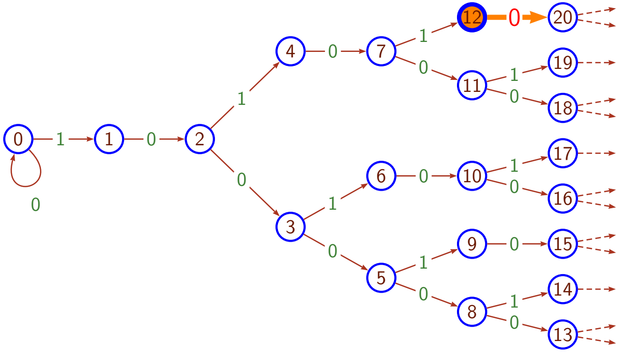

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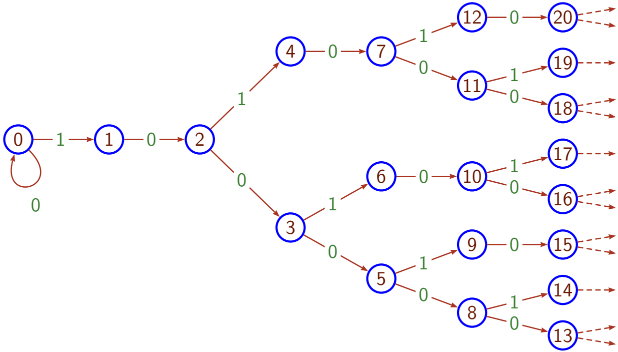

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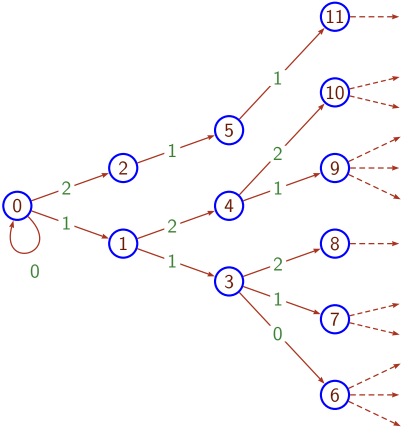
The pair (signature,labeling) is characteristic

$$\mathbf{s} = (3\ 2\ 1)^\omega$$
$$\lambda = (012\ 12\ 1)^\omega$$



Outline

Theorem (MS17a)

L: a prefix-closed language.
Signature(L) is a morphic $\Leftrightarrow$ L is a regular language.

Word Morphisms

σ: a morphism $A^* \to A^*$.

Running examples

Fibonacci morphism: $\{a, b\} \to \{a, b\}^*$

$a \mapsto ab$

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f : a letter-to-letter morphism $A^* \to B^*$.
$\to$ $f(\sigma^\omega(a))$ is called a morphic word.

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Morphic Signature

let $f_\sigma : A^* \to D^*$ be the (letter-to-letter) morphism defined by

- $D \subset \mathbb{N}$
- $\forall b,\ f_\sigma(b) = |\sigma(b)|$

We call $f_\sigma(\sigma^\omega(a))$ a **morphic signature**.

Example: Fibonacci morphism

$\sigma(a) = ab$

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\end{aligned}$$

$$\sigma(abc) \quad = \quad abc\,abc \qquad \text{hence} \quad f_\sigma(\sigma^\omega(a)) \quad = \quad (321)^\omega$$

Morphic Labeling

If g is a morphism such that

- $\forall b,\ |g(b)| = |\sigma(b)|$
- if $g(b) = c_0 c_1 \cdots c_k$ then $c_0 < c_1 < \cdots < c_k$

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If we choose g:

$g(a) = 012$

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Proof sketch

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(σ, g) defines a finite automaton $\mathcal{A}_{(\sigma,g)}$.
It is analogous to

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Proposition (MS17a)

The language accepted by $\mathcal{A}_{(\sigma,g)}$ has signature (σ, g).

Automaton associated with a morphic signature

$\sigma : \mathsf{A}^* \to \mathsf{A}^*$ prolongable on a $\qquad$ and $\qquad g : \mathsf{A}^* \to B^*$

$$\mathcal{A}_{(\sigma,g)} = \langle\, \mathsf{A}\,, B,\, \delta\,,\, \{\mathsf{a}\}\,,\, \mathsf{A}\,\rangle$$

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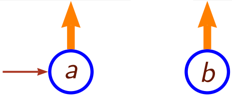

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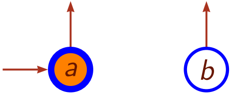

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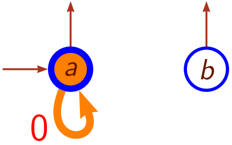

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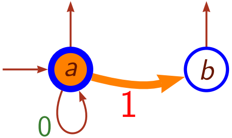

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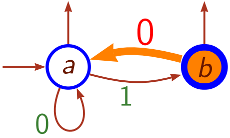

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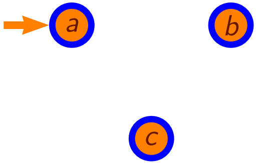

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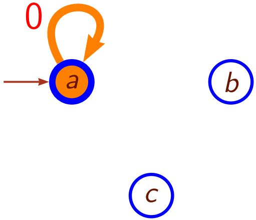

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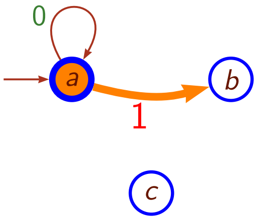

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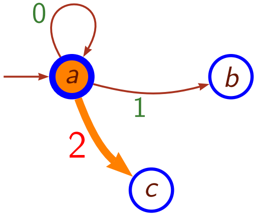

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$\sigma(c) = c$

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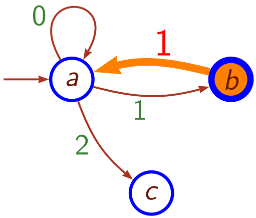

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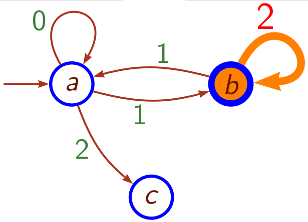

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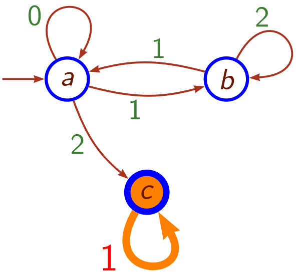

Back to ANS's

Observation

In basically every NS, the representations of integers follows the *radix order*: $$\forall n, p \quad \langle n \rangle \leq_{rad} \langle n+p \rangle$$

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Definition (ANS L)

L: language over an ordered alphabet A.
$\langle n \rangle_L$ is the $(n+1)$-th word of L in the radix order.

In our scheme, $\langle n \rangle_L$ is the word that labels the path $0 \longrightarrow n$.

For Regular ANS's, Labeling Does not Matter

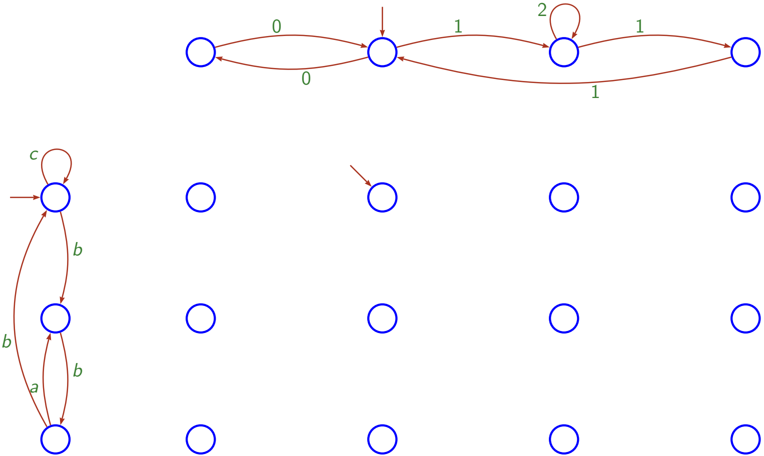
Proposition

L: regular ANS of signature (s, λ_1)
K: regular ANS of signature (s, λ_2)

The conversion function $\langle n \rangle_L \mapsto \langle n \rangle_K$ is realised by a finite, pure sequential and letter-to-letter transducer.

In other words, L and K are equivalent as NS.

Labeling Does not Matter (Idea of proof)



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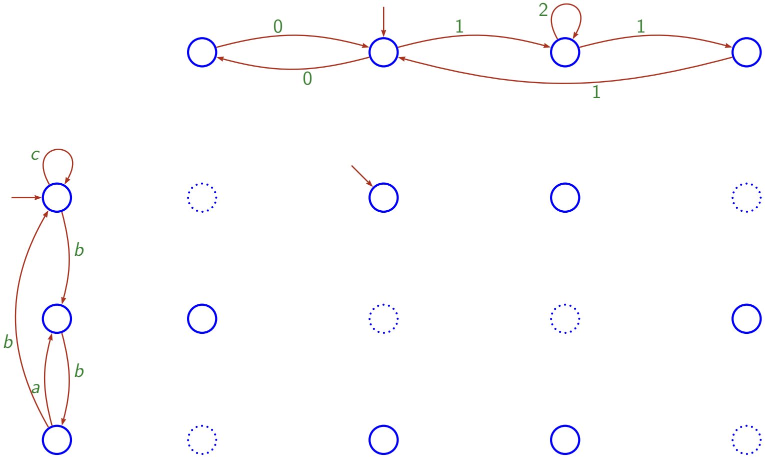

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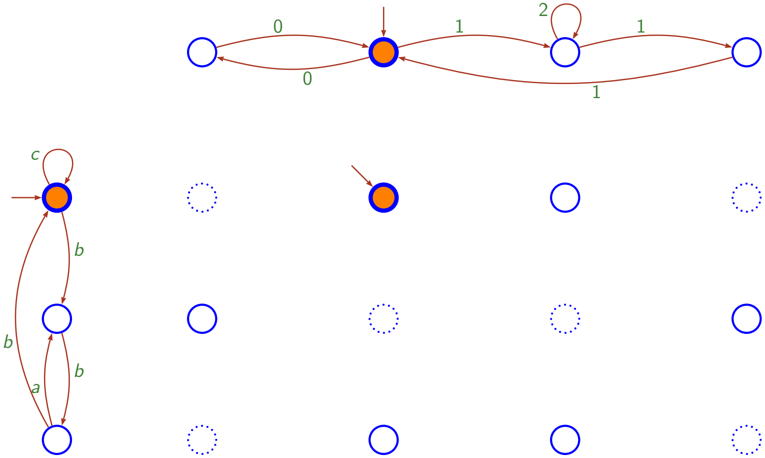

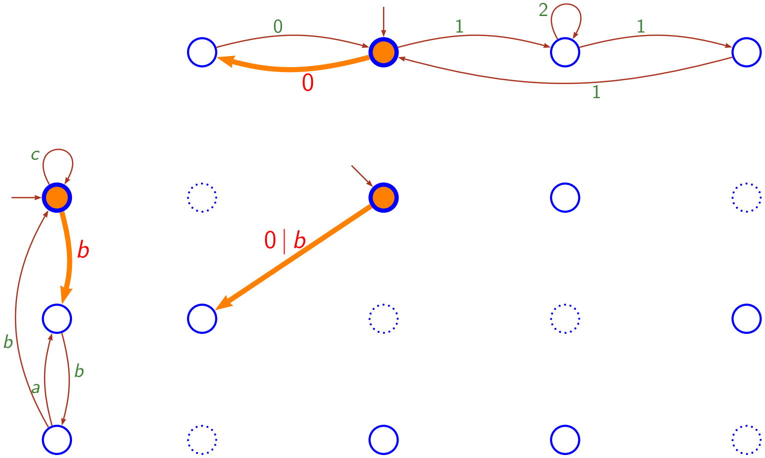
0
1
2
1
0
1
c
b
0 | b
b
b
a

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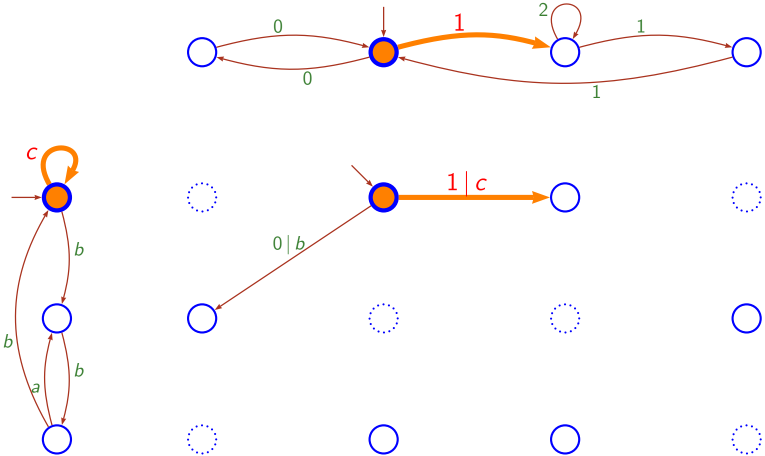

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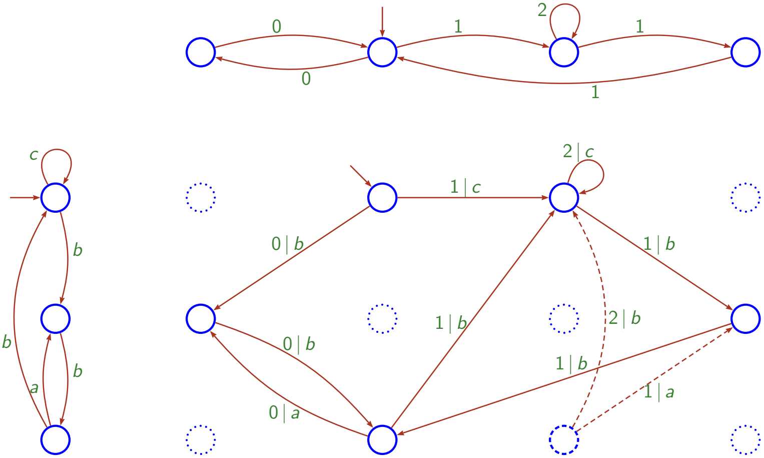

Theorem (M15)

Given a morphic signature s,
Let $C =$ the class of all regular ANS's with signature s

- In C, some regular ANS's are associated with a DFA with the minimal number of states

* Dumont and Thomas defined NS based on a pure morphic word

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Class of equivalent regular ANS's

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- C contains a Dumont-Thomas* NS
- If C contains a concrete† numeration system, then its automaton is surminimal.

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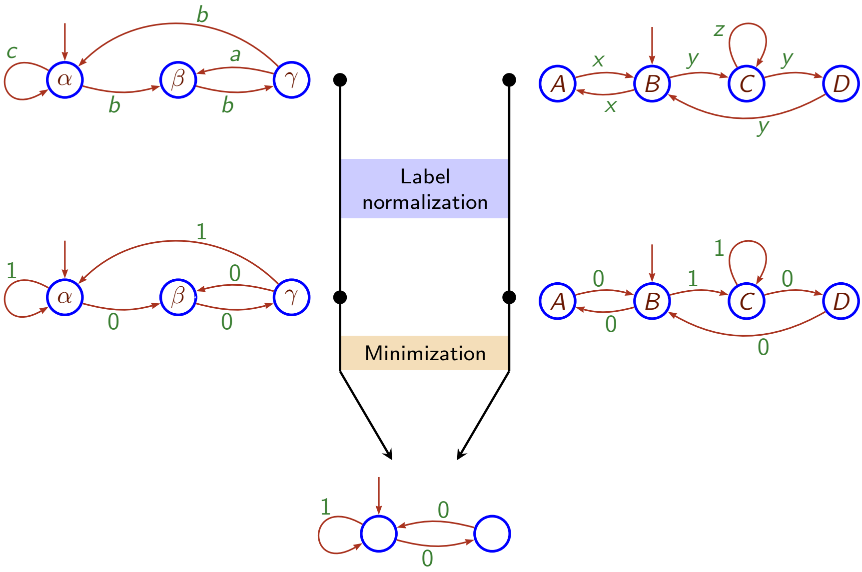
Surminimisation



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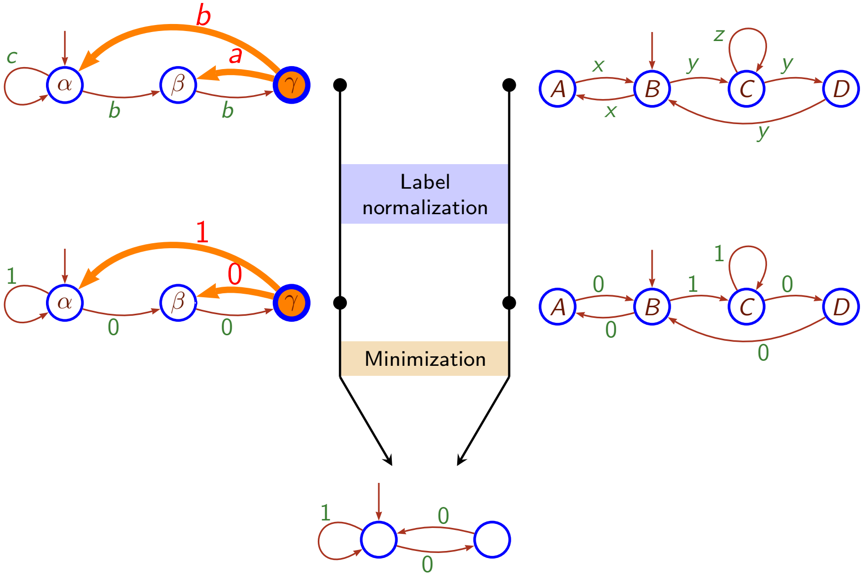

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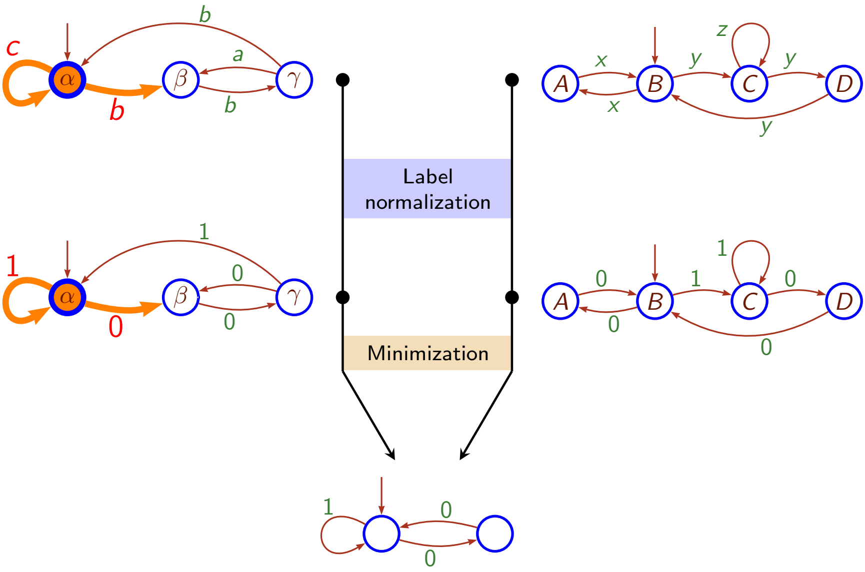

Outline

Fact

p: an integer base
L_p: representation of $\mathbb{N}$ in base p.
The language L_p has signature p^ω and Labeling $(01\cdots(p-1))^\omega$

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- base $p > 1$
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- value $\pi(a_n \cdots a_1 a_0) = \sum_{i=0}^{n} a_i p^i$

Example (base 3) -

$$\begin{aligned} \pi(12) &= \quad (3 \times 1) + (1 \times 2) \quad &= 5 \\ \pi(122) &= \quad (9 \times 1) + (3 \times 2) + (1 \times 2) \quad &= 17 \end{aligned}$$

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- value $\pi(a_n \cdots a_1 a_0) = \sum_{i=0}^{n} a_i p^i$
- $\pi(A_p^*) = \mathbb{N}$

- representation $\langle n \rangle_p = \langle n' \rangle_p . a$
  - (n', a) is the Euclidean division de n par p.
- $\langle \mathbb{N} \rangle_p = (A_p \backslash \{0\}) \cdot A_p^*$

Rational Base

- Base $\frac{p}{q} > 1$ irreducible fraction ($p > q$ and $p \wedge q = 1$).

- Representation $\langle n \rangle_{\frac{p}{q}} = \langle n' \rangle_{\frac{p}{q}}.a$:
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$2 \times 3 = 3 \times N_1 + a_0; \qquad \Rightarrow N_1 = 2$ and $a_0 = 0$.

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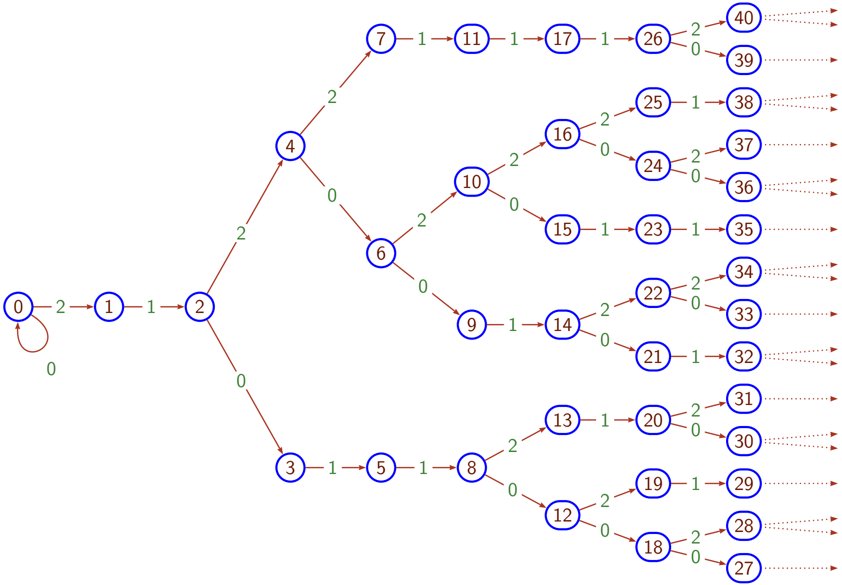
- Evaluation $\pi(a_n \cdots a_1 a_0) = \sum_{i=0}^{n} (\frac{a_i}{q})(\frac{p}{q})^i$

$L_{\frac{3}{2}}$, representation of $\mathbb{N}$ in base $\frac{3}{2}$



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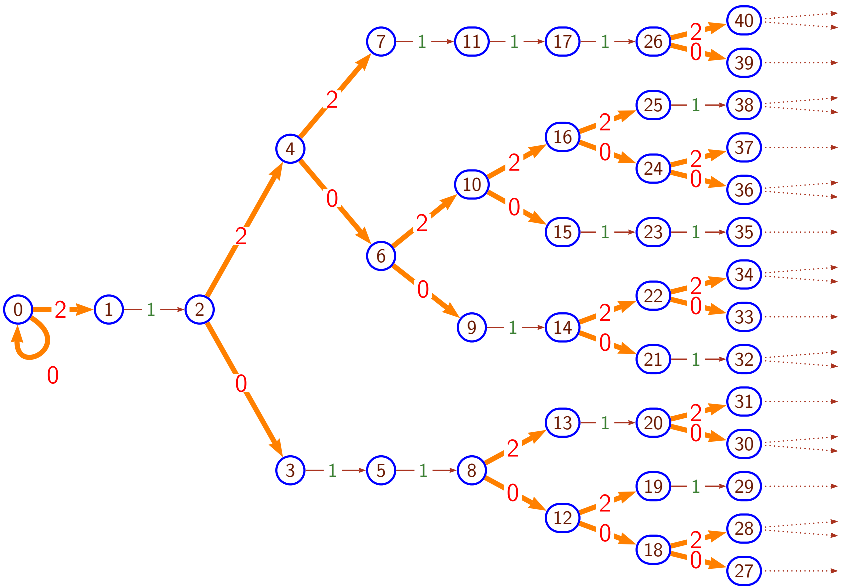
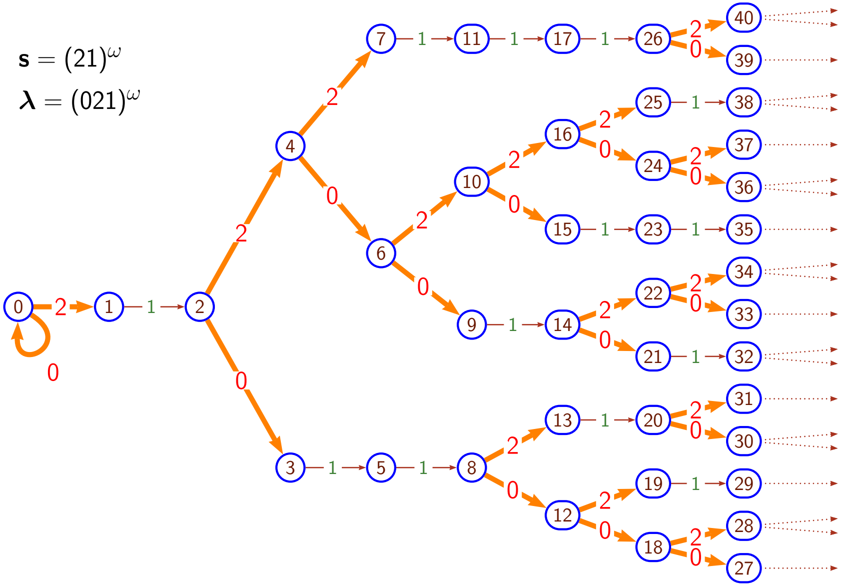


$L_{\frac{3}{2}}$, representation of $\mathbb{N}$ in base $\frac{3}{2}$

$\mathbf{s} = (21)^{\omega}$

$\boldsymbol{\lambda} = (021)^{\omega}$



Properties of $L_{\frac{p}{q}}$

- $L_{\frac{p}{q}}$ is right-extendible.
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$L_{\frac{p}{q}}$ has the Finite Left Iteration Property :

For every word u, v, $L_{\frac{p}{q}} \cap (uv^*)$ is finite

Fact

p: an integer base

L_p: representation of $\mathbb{N}$ in base p.

The language L_p has signature p^ω and Labeling $(01\cdots(p-1))^\omega$

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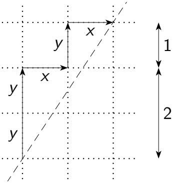
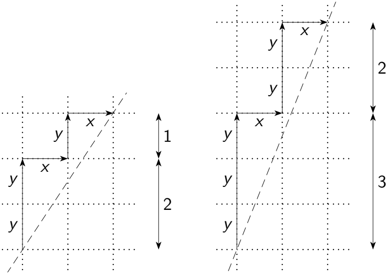


Slope $\frac{p}{q}$:	$\frac{3}{2}$
Christ. word:	$yy\,x\,y\,x$
Christ. rhythm:	$(2,\ 1)$
Sign. of $L_{\frac{p}{q}}$:	$(21)^{\omega}$

Christoffel Word and Christoffel Rhythm



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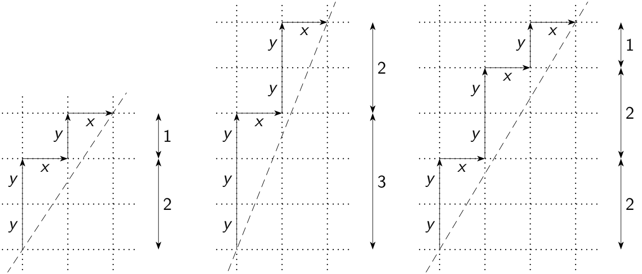
Sign. of $L_{\frac{p}{q}}$:	$(21)^\omega$	$(32)^\omega$



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Definition (Canonical Labeling)

the p-tuple: $(0, q, (2q), \ldots, (p-1)q) \quad [mod\ \ p]$

Example: $(0, 2, 1)$ for $\frac{3}{2}$ and $(0, 3, 1, 4, 2)$ for $\frac{5}{3}$.

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What About Arbitrary Periodic Signatures?

$$\text{Signature } \mathbf{s} = (s_0 s_1 \cdots s_{(q-1)})^\omega$$

- Directing parameter of s: (q, p)
  - the period length of $\mathbf{s}$ is q;
  - $p = s_0 + s_1 + s_2 + \cdots + s_{q-1}$.
- Growth ratio of $\mathbf{s}$: $\frac{p}{q}$
  - Intuition : $\#\{\text{nodes at depth } i\}$ is roughly $\left(\frac{p}{q}\right)^i$

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Theorem (MS17b)

$\exists$ *smart* labeling $\boldsymbol{\lambda} = (\lambda_0 \cdots \lambda_{p-1})^\omega$ such that
$L_{\mathbf{s},\boldsymbol{\lambda}}$, the language generated by $(\mathbf{s}, \lambda)$,
is a *noncanonical representation of* $\mathbb{N}$ in base $\frac{p}{q}$.

- If $\frac{p}{q}$ is an integer, $L_{\mathbf{s},\boldsymbol{\lambda}}$ is a regular language.
- If $\frac{p}{q}$ is not integer, $L_{\mathbf{s},\boldsymbol{\lambda}}$ is a FLIP language.

Noncanonical Representation of $\mathbb{N}$

Reminder: concrete numeration system

- Alphabet A of digits
- Evaluation function: $val : a_n \cdots a_0 \mapsto f(a_n, \ldots, a_0)$,
  where f is an arithmetic function
- Among $\{u \in A^* | \pi(u) = n\}$ one is chosen to be
  the canonical representation of n

Definition

L is a noncanonical representation of $\mathbb{N}$ in a concrete NS if

$$\forall n \in \mathbb{N}, \quad \exists u \in L, \quad \pi(u) = n$$

What About Arbitrary Periodic Signatures?

$$\text{Signature } \mathbf{s} \;=\; (s_0 s_1 \cdots s_{(q-1)})^{\omega}$$

- Directing parameter of s: (q, p)
  - the period length of **s** is q;
  - $p \;=\; s_0 + s_1 + s_2 + \cdots + s_{q-1}$.
- Growth ratio of **s**: $\frac{p}{q}$
  - Intuition : $\#\{\text{nodes at depth } i\}$ is roughly $\left(\frac{p}{q}\right)^i$

Theorem (MS17b)

$\exists$ *smart* labeling $\boldsymbol{\lambda} = (\lambda_0 \cdots \lambda_{p-1})^{\omega}$ such that
$L_{\mathbf{s},\boldsymbol{\lambda}}$, the language generated by $(\mathbf{s}, \lambda)$,
is a *noncanonical representation of* $\mathbb{N}$ in base $\frac{p}{q}$.

- If $\frac{p}{q}$ is an integer, $L_{\mathbf{s},\boldsymbol{\lambda}}$ is a regular language.
- If $\frac{p}{q}$ is not integer, $L_{\mathbf{s},\boldsymbol{\lambda}}$ is a FLIP language.

And **Eventually** Periodic Signatures?

$$\text{Signature } \mathbf{s} = x_0 \cdots x_k (s_0 s_1 \cdots s_{(q-1)})^\omega$$

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Outline

Pespectives

Two correspondance Signature $\leftrightarrow$ Numeration systems

- For regular ANS's, labeling does not matter.
  - All regular ANS's with the same signature are equivalent.
- For ANS's with ultimately periodic signature
  - labeling matters,
  - if we use the smart labeling, equivalent to some base $\frac{p}{q}$

Directed Signature (idea in M16)

Question: When does the smart labeling work?

Answer: When the signature is *directed* by $\frac{p}{q}$

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Question: When does the smart labeling work?

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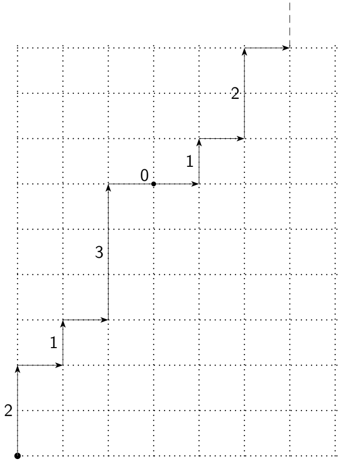
Ex: $\mathbf{s} = 2\,1\,3\,0\,1\,2\cdots$

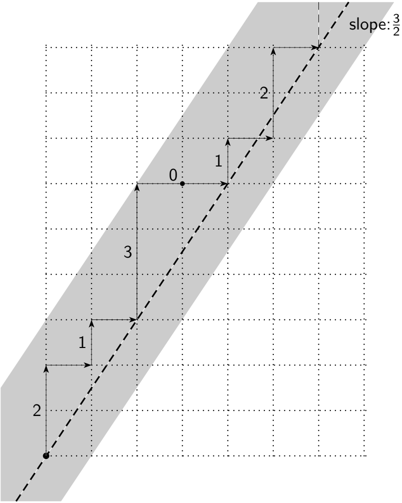


Question: When does the smart labeling work?

Answer: When the signature is *directed* by $\frac{p}{q}$

Ex: $\mathbf{s} = 2\,1\,3\,0\,1\,2\cdots$

... that is, when its path in the plane stays between two lines of slope $\frac{3}{2}$



Pespectives

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$\Rightarrow$ A regular ANS directed by $p \in \mathbb{N}$ is equivalent to base p.

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Conjecture/Future work (since 2014...)

β: a Pisot (or maybe Parry) number
S_β: the classical concrete NS based on β

A regular ANS directed by β is equivalent to S_β.