On number sets rationally represented in a rational base number system

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Outline

Integer base

- base $p \geq 2$
- alphabet $A_p = \{0, 1, \cdots, p-1\}$

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Example (base 3) - $\pi(12) = 5$ $\pi(122) = 17$

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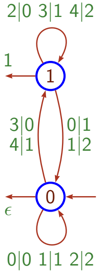
- representation $\langle n \rangle_p = a_i \cdots a_1 a_0$
  - $N_0 = n$
  - $N_k = p \times N_{(k+1)} + a_k \qquad \forall k > 0$
- $\langle \mathbb{N} \rangle_p = (A_p \backslash \{0\}) \cdot A_p^*$

Digit-wise addition : $A_p \times A_p \mapsto A_{2p-1}$

example (base 3) : 122+12 = 134

Alphabet conversion : $A_{2p-1} \mapsto A_p$



$$s \xrightarrow{a|b} t \qquad \Longleftrightarrow \qquad s + a = pt + b$$

$$\longleftarrow \; 0 \; \xleftarrow[2]{1} \; 1 \; \xleftarrow[1]{3} \; 1 \; \xleftarrow[1]{4} \; 0 \; \longleftarrow$$

Rational base

- a base $\frac{p}{q}$ with $p > q$ and p coprime with q

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Example ($\frac{3}{2}$-system) $\quad \pi(2) = 1 \quad \pi(20) = \frac{3}{2} \quad \pi(21) = 2$

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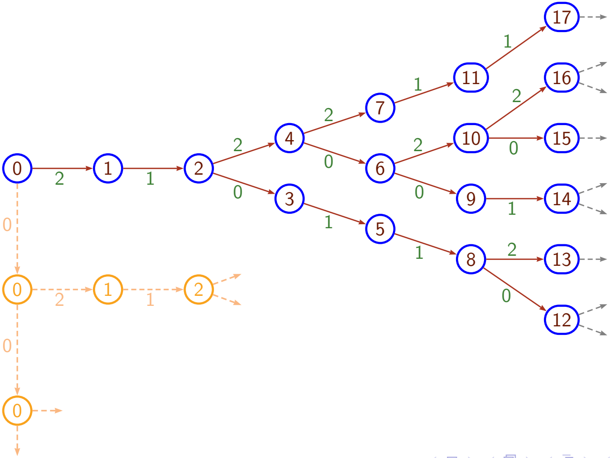
- evaluation $\pi(a_n \cdots a_1 a_0) = \sum_{i=0}^{n} (\frac{a_i}{q})(\frac{p}{q})^i$
- $\mathbb{N} \subsetneq V_{\frac{p}{q}} = \pi(A_p^*) \subsetneq \mathbb{Q}$

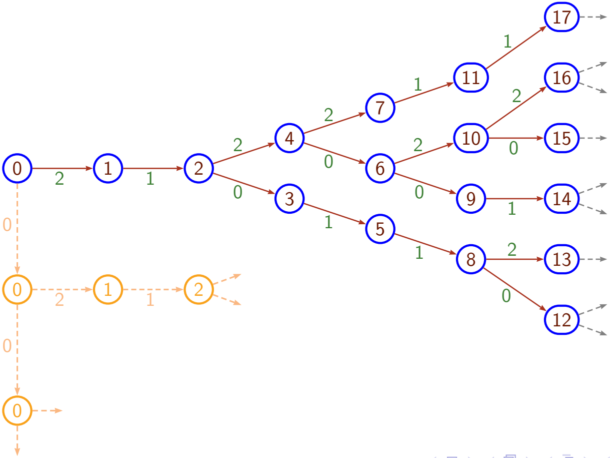
- representation $\langle n \rangle_{\frac{p}{q}} = a_i \cdots a_1 a_0$:
  - $N_0 = n$
  - $q \times N_k = p \times N_{(k+1)} + a_k \qquad \forall k > 0$
- $L_{\frac{p}{q}} = \langle \mathbb{N} \rangle_{\frac{p}{q}}$

- $L_{\frac{p}{q}}$ is prefix-closed



Rational base - $L_{\frac{p}{q}}$

- $L_{\frac{p}{q}}$ is prefix-closed
- $L_{\frac{p}{q}}$ is not rational (not even context-free) [AFS08].



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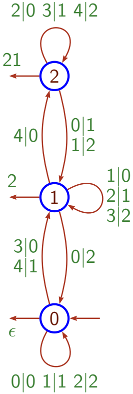
- given k, contains every number $\frac{n}{q^k}$ for n greater than some bound n_k.



$$s \xrightarrow{a|b} t \iff qs + a = pt + b$$

Problem

Is there a set simple both from

- arithmetical perspective (finitely generated)
- language theory perspective (rationally represented)

Number	Language
$V_{\frac{p}{q}}$	$\langle V_{\frac{p}{q}} \rangle = A_p^*$
$\pi(L_{\frac{p}{q}}) = \mathbb{N}$	$L_{\frac{p}{q}}$

Problem

Is there a set simple both from

- arithmetical perspective (finitely generated)
- language theory perspective (rationally represented)

The answer is NO:

Main Theorem

M finitely generated additive submonoid of $V_{\frac{p}{q}}$
$\Longrightarrow \langle M \rangle_{\frac{p}{q}}$ is not rational.

Outline

Bounded Left Iteration Property (BLIP)

A language L is BLIP if

$$\forall u, v, \quad \exists \text{ finitely many } i \quad uv^i \text{ is prefix of a word of } L$$

Example : the language $\{\epsilon, ab,\ ab.aab,\ ab.aab.aaab, \ldots\}$

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Intuition 1

- L does not contain an infinite rational language.
  [IRS : Greibach 1975]
- L is "hard" to extend to an infinite rational language.

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Intuition 2

- Every infinite branch of the tree representation of L is aperiodic

BLIP - Properties

- Every finite language is BLIP.
- Any finite union of BLIP languages is BLIP.
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- $L_{\frac{p}{q}}$ is BLIP [AFS08].

Theorem A

M finitely generated additive submonoid of $V_{\frac{p}{q}}$
$\Longrightarrow \langle M \rangle_{\frac{p}{q}}$ is a BLIP language.

Structure of a finitely generated submonoid of $V_{\frac{p}{q}}$

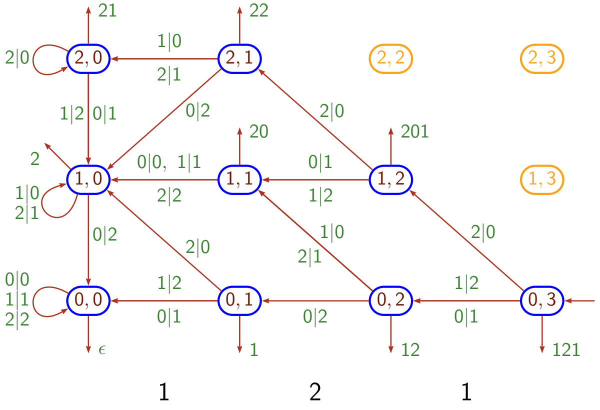
Proposition

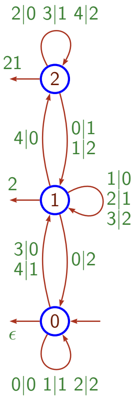
M finitely generated additive submonoid of $V_{\frac{p}{q}}$
$\Longrightarrow M \subseteq \bigcup_{i \in I} (\mathbb{N} + x_i)$ with I finite

Since BLIP is stable by sublanguage and finite union
It is enough to prove that for every $x \in V_{\frac{p}{q}}$, $\langle \mathbb{N} + x \rangle$ is BLIP.

Outline

Incrementer by 3.125 (or "121") in base $\frac{3}{2}$





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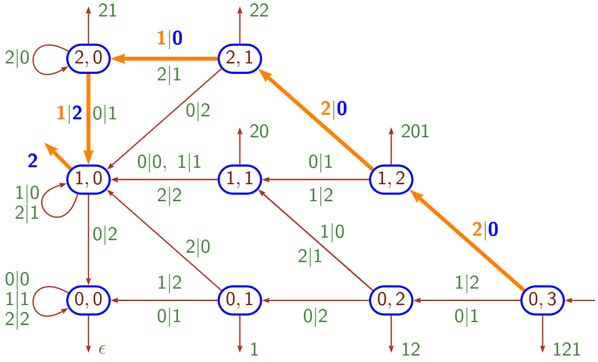

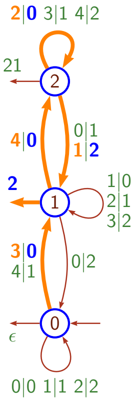

	1	2	1
1	1	2	2
22	0	0	0

	1	2	4	3
2	2	0	0	0

The incrementer and BLIP languages

Notation

$(L \oplus x) = \langle S + x \rangle$, where $L = \langle S \rangle$.

Theorem B

$L \subseteq A_p^*, \quad x \in V_{\frac{p}{q}}$

L is not BLIP $\Longrightarrow (L \oplus x)$ is not BLIP

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$(L \oplus x) = \langle S + x \rangle$, where $L = \langle S \rangle$.

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L is not BLIP $\Longrightarrow$ $(L \oplus x)$ is not BLIP

Theorem B $\Rightarrow$ Theorem A

- $\exists y \in V_{\frac{p}{q}}, (x + y) \in \mathbb{N}$
  $\Longrightarrow (\mathbb{N} + x + y) \subseteq \mathbb{N}$
  $\Longrightarrow \quad (L_{\frac{p}{q}} \oplus x \oplus y)$ is BLIP
- If $(L_{\frac{p}{q}} \oplus x)$ is not BLIP, neither is $(L_{\frac{p}{q}} \oplus x \oplus y)$

Elements of proof

L is not BLIP
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- $|w_i|$ arbitrarily large;
- all w_i reach the same state s of the incrementer by x;
- s is stable by every letter of v.

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- s is stable by every letter of v.

$$\xleftarrow{u|u'} s \xleftarrow{v^i|(v')^i} s \xleftarrow{w_i|w'_i}$$

$(L \oplus x) \ni u'(v')^i w'_i$ for infinitely many i
$\iff (L \oplus x)$ is not BLIP

M finitely generated submonoid of $V_{\frac{p}{q}}$
$\implies (M, +)$ is NOT an automatic structure.

Conclusion and future work

M finitely generated submonoid of $V_{\frac{p}{q}}$

$\Longrightarrow$ $(M, +)$ is NOT an automatic structure.

Conjecture

M additive submonoid $\mathbb{N} \subseteq M$ and $\langle M \rangle$ is rational.

$\langle M \rangle = X.A_p^*$ where $X = L_{\frac{p}{q}} \cap A_p^{\leq n}$