Ultimate periodicity of b-recognisable sets: a quasilinear procedure

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Outline

1 Introduction

2 The Pascal automaton

3 UP Criterion

4 Conclusion and Future work

Integer base

- base $b \geq 2$
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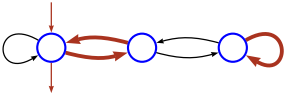
"001" "011"

$X \subseteq \mathbb{N}$ is b-rational

- automaton $\mathcal{A}$
- $L(\mathcal{A}) \xleftrightarrow{base\ b} X$

Theorem

Ultimately Periodic (UP) $\Longrightarrow$ b-rational



Example: automaton accepting integers congruent to 0 modulo 3

State of the art

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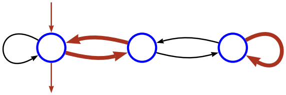

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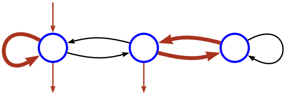

Example: automaton accepting integers congruent to 0 or 1 modulo 3

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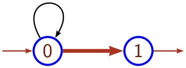
Fact

b-Rat $\not\Longrightarrow$ (UP)



Example : accepts the powers of 2

State of the art

Theorem

Ultimately Periodic (UP) $\Longrightarrow$ b-rational

Theorem (Cobham, 1969)

- X b_1-rational
- X b_2-rational
- b_1 and b_2 multiplicatively independent

$\Rightarrow X \in$ (UP)

Problem

ULTIMATE-PERIODICITY

PARAMETER :

- a base b

DATA :

- an automaton $\mathcal{A}$

OUTPUT :

- Does $L(\mathcal{A}) \in$ (UP) ?

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Theorem (Honkala, 1986)

ULTIMATE-PERIODICITY is decidable.

Two orthogonal generalisations

Theorem (Leroux, 2005)

Semi-Linear($\mathbb{N}^k$) is decidable in *b-Rat*($\mathbb{N}^k$) in P-TIME.

$\rightarrow$ Quadratic complexity

$\rightarrow$ Complicated geometrical algorithm

Two orthogonal generalisations

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Note (Allouche Shallit Rampersad, 2009)

- '$+$' is a b-rational relation.
- The class (UP) is Presburger-definable.

$\rightarrow$ Exponential complexity
$\rightarrow$ Generalisation to most common numeration systems

Contribution

Theorem

$\mathcal{A}$: a minimal automaton
It is decidable in linear time whether $L(\mathcal{A})$ is (UP).

Corollary

ULTIMATE-PERIODICITY is solvable in $O(n\,log_2(n))$ time.

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1 Introduction

2 The Pascal automaton
- Definition
- Properties

3 UP Criterion

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The Pascal automaton $\mathcal{P}_p^R$

Parameters

- (b : the base)
- p : a period, coprime with b.
- R : a set of remainders modulo p.

Expected behaviour

$u \in {A_b}^*$ accepted $\iff \pi(u) \equiv r\ [p],\ r \in R$

The Pascal automaton $\mathcal{P}_p^R$ (2)

- $\pi(ua) = \pi(u) + a\, b^{|u|}$
- let ψ be the smallest integer s.t. $b^{\psi} \equiv 1\ [p]$
- $b^k \equiv b^{(k \bmod \psi)} [p]$

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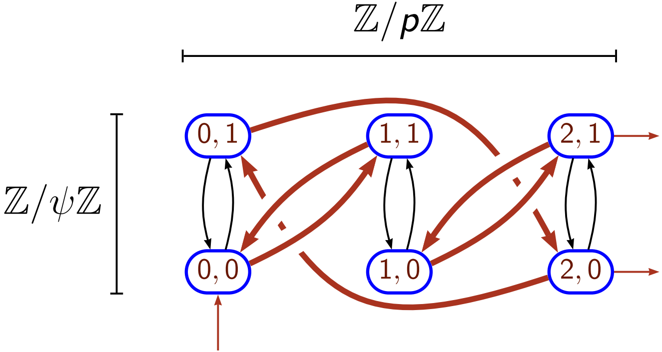
- States: $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}$

  $\pi(u) \bmod p \to \mathbb{Z}/p\mathbb{Z}$; $|u| \bmod \psi \to \mathbb{Z}/\psi\mathbb{Z}$
- Transitions: $(r, s) \xrightarrow{a} (r + ab^s, s + 1)$
- Initial state: $(0, 0)$
- Final states: $R \times \mathbb{Z}/\psi\mathbb{Z}$

Example: $\mathcal{P}_3^{\{2\}}$

- $(b = 2)$
- $p = 3$
- $\psi = 2$ (since $2^2 \equiv 1\ [3]$)



Lemma

$\mathcal{P}_p^R$ is deterministic and co-deterministic.

→ Each letter induces a permutation of the states

→ The syntactic monoid of $\mathcal{P}_p^R$ is a group $(\simeq \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/\psi\mathbb{Z})$.

Structural properties

Lemma

$\mathcal{P}_p^R$ is deterministic and co-deterministic.

Proposition

The syntactic monoid of $\mathcal{P}_p^R$ is generated by 0 and 1

or 0 and $g = 10^{-1}$

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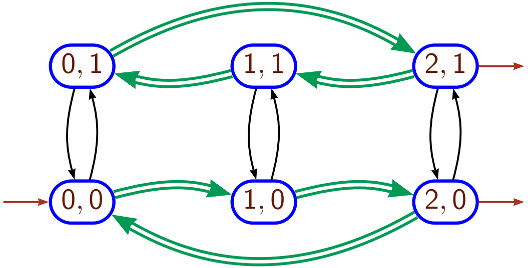

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Lemma (isotropism)

Changing the initial state of a Pascal automaton $\mathcal{P}_p^R$ yields $\mathcal{P}_S^p$ for some S.

Theorem

Given an automaton $\mathcal{A}$,
it is decidable in linear time whether $\mathcal{A}$ is the quotient of a Pascal.

Recognising a quotient of a Pascal automaton (1)

$\mathcal{P}_p^R$: a Pascal automaton
$\mathcal{A}$: a quotient of $\mathcal{P}_p^R$
$\sim$: the equivalence relation of the quotient

Step 1 – Simplifications

Changing the alphabet of $\mathcal{A}$: from $\{0, 1, \ldots, b-1\}$ to $\{0, g\}$.

Recognising a quotient of a Pascal automaton (1)

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$\mathcal{A}$: a quotient of $\mathcal{P}_p^R$
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Step 1 – Simplifications

Changing the alphabet of $\mathcal{A}$: from $\{0, 1, \ldots, b-1\}$ to $\{0, g\}$.

Step 2 – Computation of the parameters

g induces in $\mathcal{A}$ only cycles of length p.
$\rightarrow$ Yields p then indirectly R, ψ and
t, the smallest second component among state $\sim (0, 0)$.

Recognising a quotient of a Pascal automaton (2)

Step 3 – Verification

In every $\sim$-equivalence class, there is exactly one state of the form (s', t') with $t' < t$.

Browse the automaton $\mathcal{A}$ marking the states:

- The initial state is marked as (0,0)
- If a state is marked (x,y), then
  - Its successor by g is marked $(x + b^y,\ y)$
  - Its successor by 0 is marked $(x,\ y + 1)$ if $y + 1 < t$

    or
  - Its successor by 0 must be $(\frac{x-s}{b^t}, 0)$ otherwise

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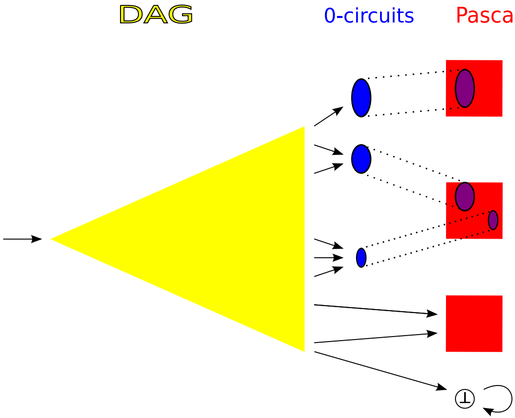
3 UP Criterion
- Statement
- Completeness
- Correctness

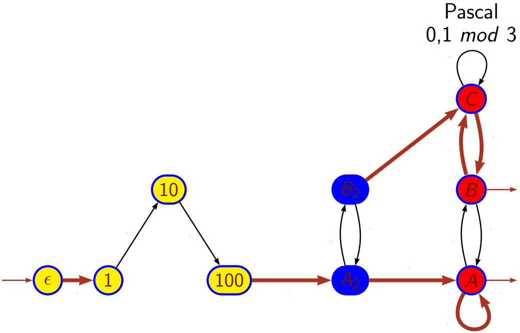
4 Conclusion and Future work

UP-Criterion

SCC: maximal Strongly Connected Component



(UP) Criterion – Example



Theorem

It can be verified in linear time whether a given automaton satisfies the UP-Criterion.

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Theorem

$\mathcal{A}$: a *minimal* automaton.
$\mathcal{A}$ satisfies the UP-criterion $\iff$ $L(\mathcal{A})$ is (UP).

Every UP-automaton satifies the criterion

Proposition 1

The UP-criterion is stable by quotient.

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Proposition 2

Every set of (UP) is recognized by an automaton satisfying the UP-criterion.

Hence, since the minimal quotient is unique:

Theorem (Completeness)

The minimal automaton accepting a given (UP) set satifies the UP-criterion.

Hint on Proposition 2

a set of (UP) : $\{n \mid n > \text{m} \text{ and } n \equiv r\ [\text{p}] \text{ with } r \in \text{R}\}$

- preperiod (m)
- period (p)
- remainder set (R)

Hint on Proposition 2

a set of (UP) : $\{n \mid n > \text{m} \text{ and } n \equiv r\ [\text{p}] \text{ with } r \in \text{R}\}$

- preperiod
- period $\Rightarrow$ Pascal's period & DAG size.
- remainder set

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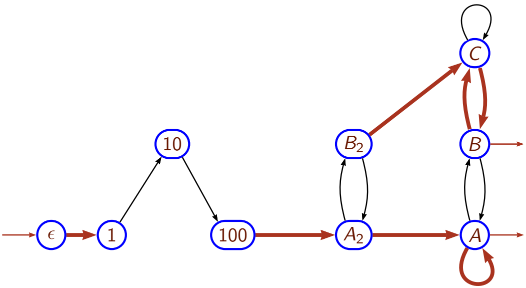
- preperiod $\Rightarrow$ 0-circuits & DAG size.
- period $\Rightarrow$ Pascal's period & DAG size.
- remainder set $\Rightarrow$ # of Pascal's & Pascal's remainders

Correctness on an example



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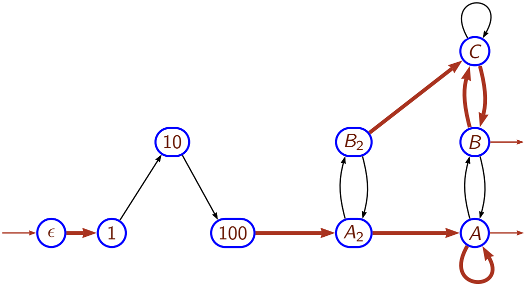

$L = \{\ u \mid\ u \text{ starts with } 10010^n1 \text{ and } \pi(u) \equiv 0,1\ [3]\ \}$

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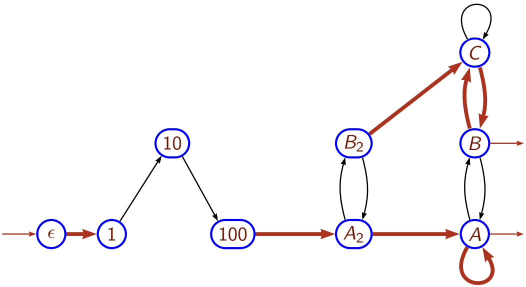

$$L = \{\ u \mid \underbrace{u \text{ starts with } 10010^n1}_{\iff \pi(u) \equiv 9\ (16) \text{ and } u > 16} \text{ and } \pi(u) \equiv 0{,}1\ [3]\ \}$$

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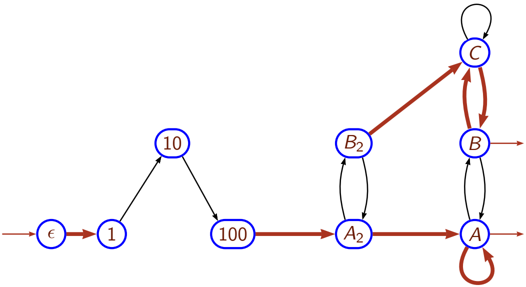

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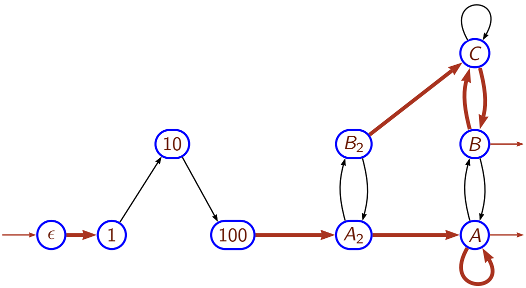

$$L = \{\ u \mid \pi(u) \equiv 9, 25\ [48] \text{ and } \pi(u) > 16.\ \}$$

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Conclusion

- Quasilinear algorithm to decide whether a DFA is (UP)
- Structural characterisation of minimal (UP) DFA

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Future work

- Getting rid of the minimality condition
  → Work in progress...
- Getting rid of determinism condition
  → Seems unrealistic with this method.
- Generalising this method to U-Systems
  → The "*isotropism* lemma" is false in the general case
  → Yields an EXP-TIME algorithm (no better than [ASR'09])