On number sets rationally represented in a rational base number system

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Outline

Integer base

- base $p \geq 2$
- alphabet $A_p = \{0, 1, \cdots, p-1\}$

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Example (base 3) - $\pi(12) = 5$ $\pi(122) = 17$

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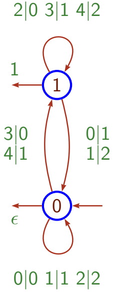
- representation $\langle n \rangle_p = a_i \cdots a_1 a_0$ (greedy algorithm).
- $\langle \mathbb{N} \rangle_p = (A_p \backslash \{0\}) \cdot A_p^*$

Integer base - additionner

Digit-wise addition : $A_p \times A_p \mapsto A_{2p-1}$

example (base 3) : $122+12 = 134$

Alphabet conversion : $A_{2p-1} \mapsto A_p$



$$s \xrightarrow{a|b} t \iff s + a = pt + b$$

$$\longleftarrow\ 0\ \xleftarrow[2]{1}\ 1\ \xleftarrow[1]{3}\ 1\ \xleftarrow[1]{4}\ 0\ \longleftarrow$$

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Example ($\frac{3}{2}$-system) $\pi(2) = 1$ $\pi(20) = \frac{3}{2}$ $\pi(21) = 2$

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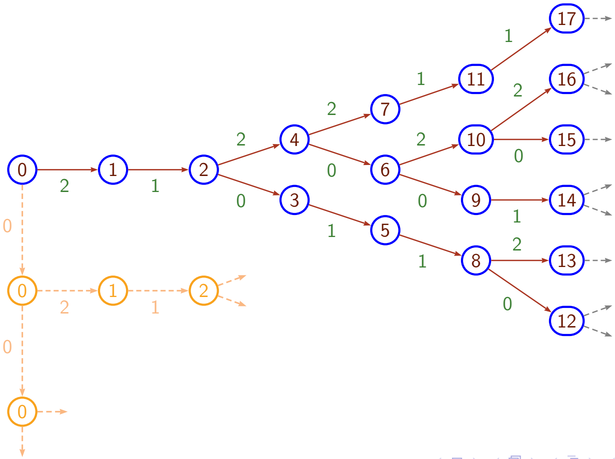
- $L_{\frac{p}{q}} = \langle \mathbb{N} \rangle_{\frac{p}{q}}$

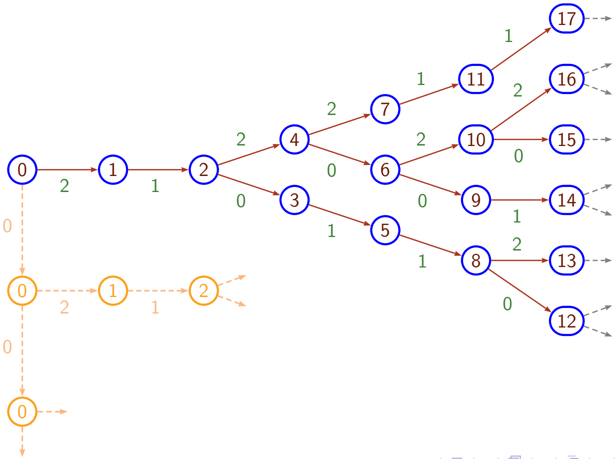
Rational base - $L_{\frac{p}{q}}$

- $L_{\frac{p}{q}}$ is prefix-closed



Rational base - $L_{\frac{p}{q}}$

- $L_{\frac{p}{q}}$ is prefix-closed
- $L_{\frac{p}{q}}$ is not rational (not even context-free) [AFS08].



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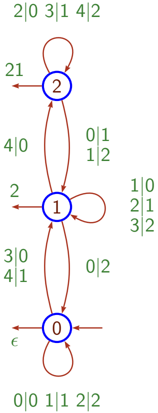
- given k, contains every number $\frac{n}{q^k}$ for n greater than some bound n_k.

Rational base - Additioner



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Is there a set simple both from

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The answer is NO:

Main Theorem

M finitely generated additive submonoid of $V_{\frac{p}{q}}$
$\Longrightarrow \langle M \rangle_{\frac{p}{q}}$ is not rational.

Outline

A language L is BLIP if
for all words u and v, there exists i such that uv^i is prefix of no word of L.
Example : the language $\{\epsilon, ab,\ ab.aab,\ ab.aab.aaab, \ldots\}$

Bounded Left Iteration Property (BLIP)

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Intuition

- L does not contain an infinite rational language.
  [IRS : Greibach 1975]
- L is "hard" to extend to an infinite rational language.

BLIP - Properties

- Every finite language is BLIP.
- Any finite union of BLIP languages is BLIP.
- Any intersection of BLIP languages is BLIP.
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- $L_{\frac{p}{q}}$ is BLIP [AFS08].

Theorem A

M finitely generated additive submonoid of $V_{\frac{p}{q}}$
$\implies \langle M \rangle_{\frac{p}{q}}$ is a BLIP language.

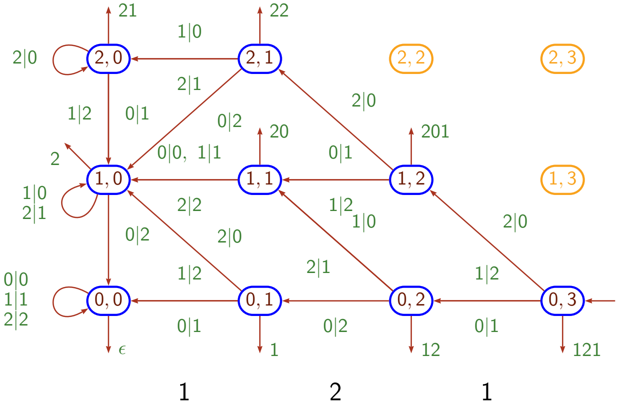
Structure of a finitely generated submonoid of $V_{\frac{p}{q}}$

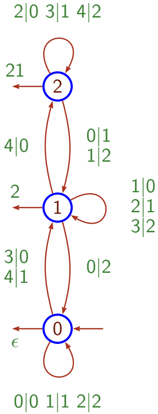
Proposition

M finitely generated additive submonoid of $V_{\frac{p}{q}}$
$\Longrightarrow M \subseteq \left(\bigcup_i (\mathbb{N} + x_i)\right)$

Outline

Incrementer by 3.125 (or "121") in base $\frac{3}{2}$





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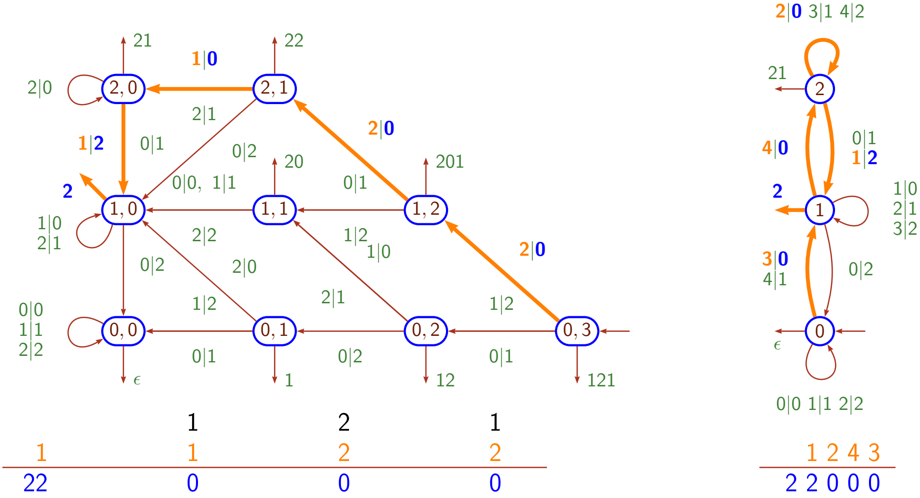

Theorem B

$L \subseteq A_p^*, \quad x \in V_{\frac{p}{q}}$

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Theorem B $\Rightarrow$ Theorem A

- $\exists y \in V_{\frac{p}{q}}, (x+y) \in \mathbb{N}$
  $\Longrightarrow (\mathbb{N} + x + y) \subseteq \mathbb{N}$
  $\Longrightarrow \quad (L_{\frac{p}{q}} \oplus x \oplus y)$ is BLIP
- If $(L_{\frac{p}{q}} \oplus x)$ is not BLIP, neither is $(L_{\frac{p}{q}} \oplus x \oplus y)$

Elements of proof

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WLOG

- $|w_i|$ arbitrarily large;
- all w_i reach the same state s of the incrementer by x;
- s is stable by every letter of v.

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- s is stable by every letter of v.

$$\xleftarrow{u|u'} s \xleftarrow{v^i|(v')^i} s \xleftarrow{w_i|w'_i}$$

$(L \oplus x) \ni u'(v')^i w'_i$ for infinitely many i
$\iff (L \oplus x)$ is not BLIP

Conclusion and future work

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$\implies (M, +)$ is NOT an automatic structure.

Conclusion and future work

M finitely generated submonoid of $V_{\frac{p}{q}}$
$\Longrightarrow$ $(M, +)$ is NOT an automatic structure.

Conjecture

M additive submonoid $\mathbb{N} \subseteq M$ and $\langle M \rangle$ is rational.
$\langle M \rangle = X.A_p^*$ where $X = L_{\frac{p}{q}} \cap A_p^{\leq n}$